Outlier Removal for Prediction of Covariance Matrices
With an Application to Portfolio Optimization

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Overview of the talk

- Portfolio theory
- Predictions of Covariance matrices and return vectors
- The Naive Prediction
- An algorithm to remove outliers

Modern Portfolio Theory

Harry Markowitz, William Sharpe

- Investors hold portfolios of assets and are therefore focused on the return and risk for the whole portfolio, not for individual assets.
- The risk is quantified by the standard deviation for the portfolio.
- For a given expected return we can get different expected standard deviation depending on the mix of assets (due to varying correlations between the assets).
- The optimal portfolio can be determined by solving a quadratic optimisation problem.

Modern Portfolio Theory

- The stock prices are r.v. with returns $r_1, \ldots, r_n$ with :
  Expected values $\mu_1, \ldots, \mu_n \equiv R$
  Covariances $\sigma_{11}, \sigma_{12} \ldots, \sigma_{nn} \equiv C$
- The portfolio is defined by the weights $\mathbf{w} = w_1, \ldots, w_n$ and has:

a return $R_p = \sum_{i=1}^{n} w_i r_i$

with expectation $\mu_p \equiv ER_p = \sum_{i=1}^{n} w_i Er_i = \sum_{i=1}^{n} w_i \mu_i = w^T R$

and a variance $\sigma_p^2 = E(R_p - ER_p)^2 = E\left(\sum_{i=1}^{n} w_i r_i - \sum_{i=1}^{n} w_i \mu_i\right)^2 =$

$$\sum_{i=1}^{n} w_i^2 \sigma_i^2 + 2\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} w_i w_j \rho_{ij} \mu_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} =$$

$$w^T C\ w$$

Modern Portfolio Theory

- The risk adjusted return R_{ADJ} is defined as

  $\alpha R_p - \sigma_p^2 = \alpha w^T R - w^T C w$

  The risk tolerance factor α expresses the relative importance of the risk and return (often set to 0.5)

- Optimization problem :

$$\max_{w} \alpha w^T R - w^T C w$$
s.t.:
$$w_i \quad 0,\ i=1,\ldots,n$$
$$\Sigma w_i = 1$$

Procedure for Portfolio Optimization

1. Estimate with historical data:
   - The individual returns $\mu_1, \ldots, \mu_n \equiv R$
   - The covariance matrix C
2. Solve
   $w_{OPT} = \max_{w} \alpha w^T R - w^T C w$
   for "optimal" portfolio weights w_{OPT}
3. Rebalance the portfolio

Repeat from 1. every n:th day.

Question: How optimal are the weights ?

How optimal are the weights

Answer: It depends on the stationarity of the price processes

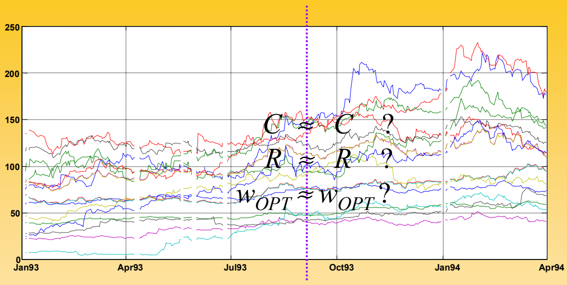
$C \approx C$?

$R \approx R$?

$w_{OPT} \approx w_{OPT}$?

Estimation of C and R and optimization | Evaluation: Computation of R_{ADJ} | *time*

How Stationary are the Processes ?



Estimation of R and C and optimization | Evaluation: Computation of R_{ADJ} | *time*

We have to PREDICT covariances C and returns R

Common methods:

1. The “Naive prediction”:
   The sample covariances and returns in a window backwards.
2. Exponentially Weighted Moving Average
3. ARCH, GARCH, D-GARCH ….

.
.
.

The Naive Prediction

At time T:
Use stock prices $Close_i(t), i=1,\ldots,n \quad t\in \{T-w,\ldots,T-1\}$
to compute sample covariances and returns:

$$\mu_j^* = \frac{1}{w}\sum_{\tau=T-w}^{T-1} Close_j(\tau)$$

$$\sigma_{ij}^* = \frac{1}{w-1}\sum_{\tau=T-w}^{T-1}(Close_i(\tau)-\mu_i^*)(Close_j(\tau)-\mu_j^*)$$

Assume that these values are valid for $t\in \{T,\ldots,T+v\}$ and optimize away

The Naive Prediction

How much past data (w) should we use ?

Empirical investigation:

w days for estimation of R and C, optimization and computation of R_{ADJ} | 20 days for estimation of R and C, optimization and computation of R_{ADJ} | *time*

For w = 20,40,60,…,300 :
Sliding window testing with two data sets with data from 1988-1997 :
24 Swedish stocks
29 American stocks

Sliding Windows

Necessary since we can’t use cross validation

time

w days for modelling | v days for evaluation

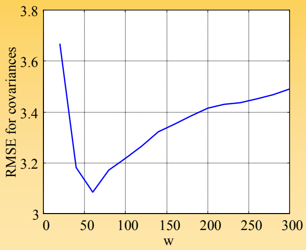
Prediction 1
Prediction 2
Prediction 2
.
. v days step
.
Prediction K

The performance is computed as the average performance for the K predictions

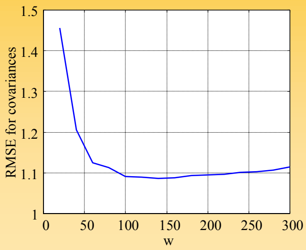
The Naive Prediction

How much past data should we use ?

Swedish stocks 88-97



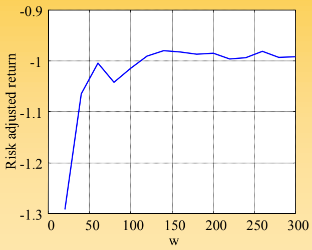
American stocks 88-97



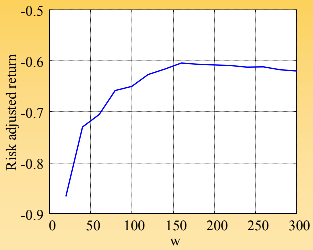
The Naive Prediction

200 days looks like a good choice

Swedish stocks 88-97



American stocks 88-97



Improving the Naive Prediction

Assumption: The Naive Prediction gains from removing certain days which can be regarded as Outliers or Noise.
The "Outlier" property for a day generalises into the future.
I.e. The same day should be removed in the next step.

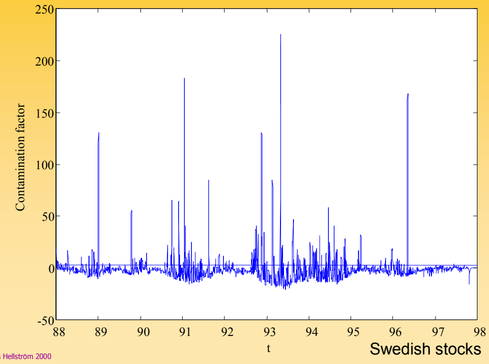
Idea:
For each step in the sliding window predictions:
    Compute a prediction where all days t with "Contamination Factor" $K_t > K_{limit}$ have been removed
    For each day t in the modelling data window:
        Remove day t, build models and predict C and R.
        Compare the prediction with one without removals.
        Update K_t depending on this comparision.
    next day t
next step

The *Remove* Algorithm:

$K_i \leftarrow 0, i = d_1, ..., d_2$
for $t = d_1 + w$ step v to $d_2 - v$
    $r \leftarrow \{j | K_j > K_{limit}, j \in (t-w, ..., t-1)\}$
    $\hat{C}(t) \leftarrow cov(t-w, t-1, r)$
    $C(t) \leftarrow cov(t, t+v-1)$
    $\hat{C}_0(t) \leftarrow cov(t-w, t-1)$
    $e_0 \leftarrow \left\| C(t) - \hat{C}_0(t) \right\|_2^2$
    for $j = t-w$ to $t-1$
        $\hat{C}_j(t) \leftarrow cov(t-w, t-1, j)$
        $e_j \leftarrow \left\| C(t) - \hat{C}_j(t) \right\|_2^2$
        $K_j \leftarrow K_j + \frac{e_0 - e_j}{e_0} 100$
    next j
next t

The Contamination Factors K_t

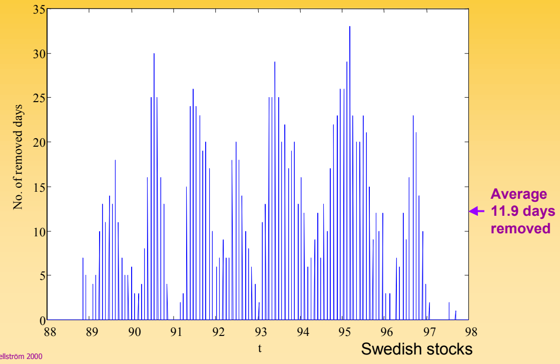
Large K_t means that day t should be removed when calculating C and R



Number of Removed Days

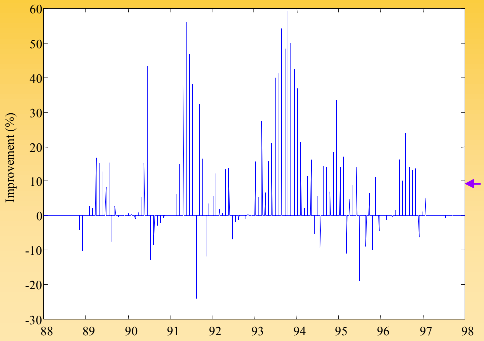
for each monthly prediction $\hat{C}(t)$.

A day j is removed iff $K_j > K_{limit}$ (K_{limit}=3 in the examples)



Improvement Relative the Naive Prediction

Improvement measured as RMSE
for the covariance predictions



Average Improvement 8.9%

Swedish stocks

The Contamination Factors K_t

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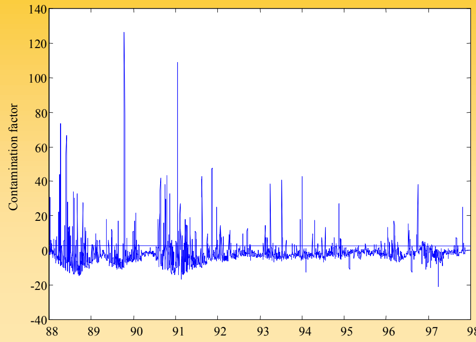

American stocks

Number of Removed Days

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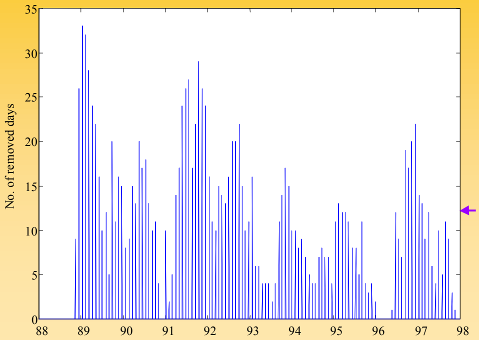

Average 11.7 days removed

American stocks

Improvement Relative the Naive Prediction

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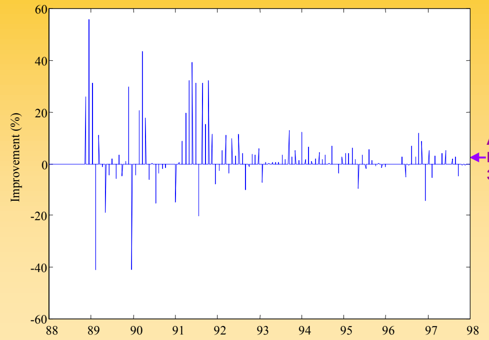


Average Improvement 3.5%

American stocks

Results from Portfolio Optimization

Swedish stocks 88-97 :

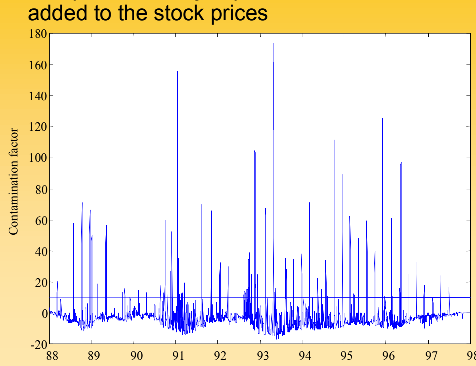
Method	R_p	σ_p^2	R_{ADJ}	Best
$\hat{C}_j$: Remove 3%	0.067	0.995	-0.959	62%
$\hat{C}_0$: Naive 200 days	0.065	1.019	-0.984	47%
Improvement	3.2%	2.4%	2.5%	
Equally balanced	0.072	1.465	-1.428	

American stocks 88-97 :

Method	R_p	σ_p^2	R_{ADJ}	Best
$\hat{C}_j$: Remove 3%	0.061	0.627	-0.597	54%
$\hat{C}_0$: Naive 200 days	0.055	0.636	-0.608	51%
Improvement	10.5%	1.4%	1.9%	
Equally balanced	0.065	0.727	-0.694	

Artificial Data

Every 50th trading day has a random noise
added to the stock prices



The noisy days are clearly detected and removed by the algorithm

Summary and Conclusions

- The RMSE for covariances is reduced by:
  8.9% for the Swedish data
  3.5% for the American data
- The increase in Risk Adjusted Return R_{ADJ} is:
  2.5% for the Swedish data
  1.9% for the American data
- Conclusion: Outliers don't affect the computation of optimal portfolios very much ?
- The *Remove* algorithm successfully detects and removes outliers in data.
- Ref: Outlier Removal for Prediction of Covariance Matrices with an Application to Portfolio Optimization
  Thomas Hellström. Technical report UMINF 00.19 Department of Computing Science, Umeå University