Robust Computation of Canonical Structure Information of Matrices and Matrix Pencils - from Staircase Algorithms to Orbit and Bundle Stratifications

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Theory and Numerics of Matrix Eigenvalue Problems
Nov. 22-27, 2003 - Banff Intern. Research Station, Canada

Matrix

Matrix Reloaded

Matrix Revolutions

Matrix Stratifications

www.cs.umu.se/research/nla/singular_pairs

Coming soon to a PC near you!

Control System Design and Analysis

- Consider $x'(t) = Ax(t) + Bu(t)$
  - A's Jordan structure reveals the dynamics of the system
  - $[A - \lambda I \mid B]$'s Kronecker structure reveals its controllability properties
- Computing canonical forms are ill-posed - rely on Schur-like forms: $Q^*(A+E)Q = T$
- What can we say about other nearby canonical structures (e.g., distance to uncontrollability)?

Degeneration via confluence

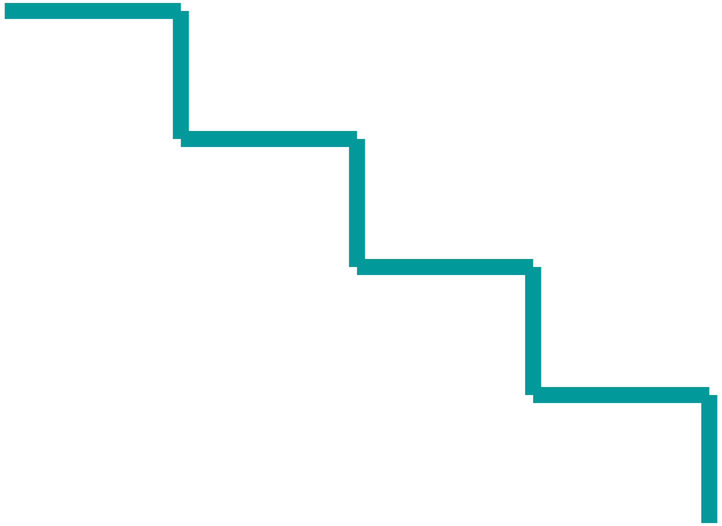
- Computing the JNF is an ill-posed problem when we have “multiple” or “ill-conditioned” eigenvalues
  - small perturbations to input data may drastically change the computed Jordan structure
- Schur forms of 4-by-4 matrices

$$\begin{bmatrix} \alpha & x & x & x \\ 0 & \alpha & x & x \\ 0 & 0 & \alpha & x \\ 0 & 0 & 0 & \alpha \end{bmatrix} \begin{bmatrix} \alpha & x & x & x \\ 0 & \alpha & x & x \\ 0 & 0 & \alpha & x \\ 0 & 0 & 0 & \beta \end{bmatrix} \begin{bmatrix} \alpha & x & x & x \\ 0 & \alpha & x & x \\ 0 & 0 & \beta & x \\ 0 & 0 & 0 & \beta \end{bmatrix} \begin{bmatrix} \alpha & x & x & x \\ 0 & \alpha & x & x \\ 0 & 0 & \beta & x \\ 0 & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} \alpha & x & x & x \\ 0 & \beta & x & x \\ 0 & 0 & \gamma & x \\ 0 & 0 & 0 & \delta \end{bmatrix}$$

Confluence warning!

In case of fire,
do not use the elevator,

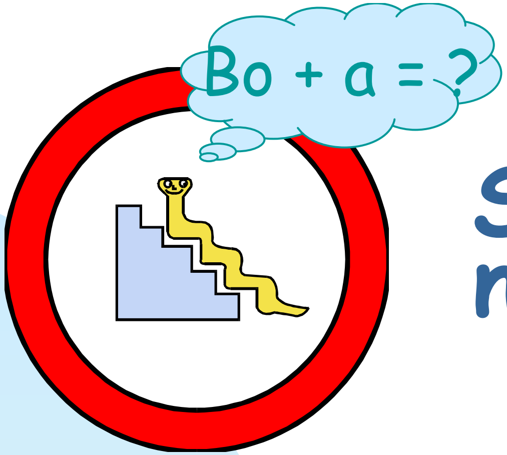
… use the stairs!



The staircase method

- Determine the canonical structure of a multiple eigenvalue
  - Regularization - “well-posed problem”
- Vera Kublanovskaya (1966)
  - Honory Doctor at Umeå Univ. (1985)
- Analyzed in finite precision arithmetic by Axel Ruhe (1970):

$$Q^*(A+E)Q = T$$



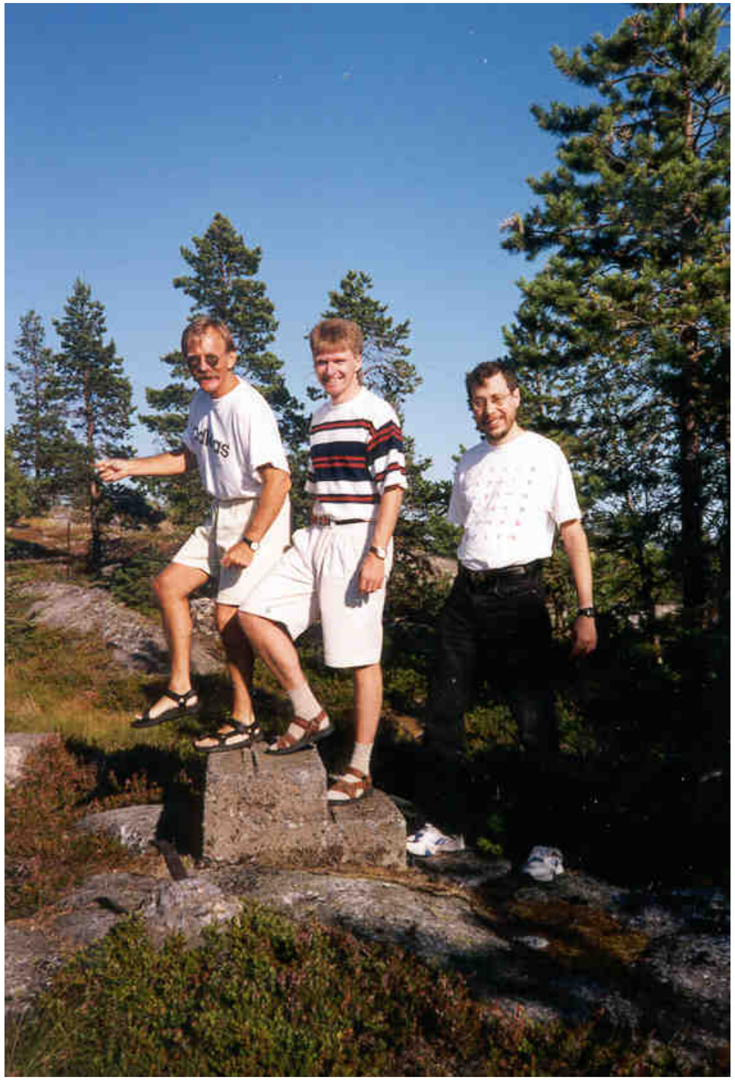
Staircase form - 7x7 nilpotent

$$\begin{array}{ccc|cc|cc}
\multicolumn{3}{c}{\overbrace{\qquad\qquad\quad}^{m_1}} & \multicolumn{2}{c}{\overbrace{\qquad\quad}^{m_2}} & \multicolumn{2}{c}{\overbrace{\qquad\quad}^{m_3}} \\
\left[\begin{array}{ccc} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{array}\right. & & & & & & \\
\end{array}$$

$$\left[\begin{array}{ccc|cc|cc}
0 & 0 & 0 & x & x & x & x \\
 & 0 & 0 & x & x & x & x \\
 & & 0 & x & x & x & x \\
\hline
 & & & 0 & 0 & y & y \\
 & & & & 0 & y & y \\
\hline
 & & & & & 0 & 0 \\
 & & & & & & 0
\end{array}\right]$$

$$\begin{aligned}
m_1 &= 3 = \dim \mathcal{N}(A - \mu I), \\
m_1 + m_2 &= 5 = \dim \mathcal{N}((A - \mu I)^2) \\
m_1 + m_2 + m_3 &= 7 = \dim \mathcal{N}((A - \mu I)^3)
\end{aligned}$$

A step into the unknown!



Some matrix space geometry

- View an nxn *A* as a point in n^2-dim space

- *Orbit(A):* the manifold of all matrices with the Jordan Normal Form (JNF) of *A*:

  $$\text{orbit}(A) \equiv \{PAP^{-1}: \det(P) \neq 0\}$$

- *Bundle(A):* the union of all orbits with the JNF of *A* but with eigenvalues different

- The *codimension* of an orbit (bundle) is the dimension of the space complementary to the orbit (bundle), i.e., dim(nor(*A*))

Matrix Space Stratifications

- Given an nxn matrix *A* and its orbit:

What other canonical structures are found within its closure?

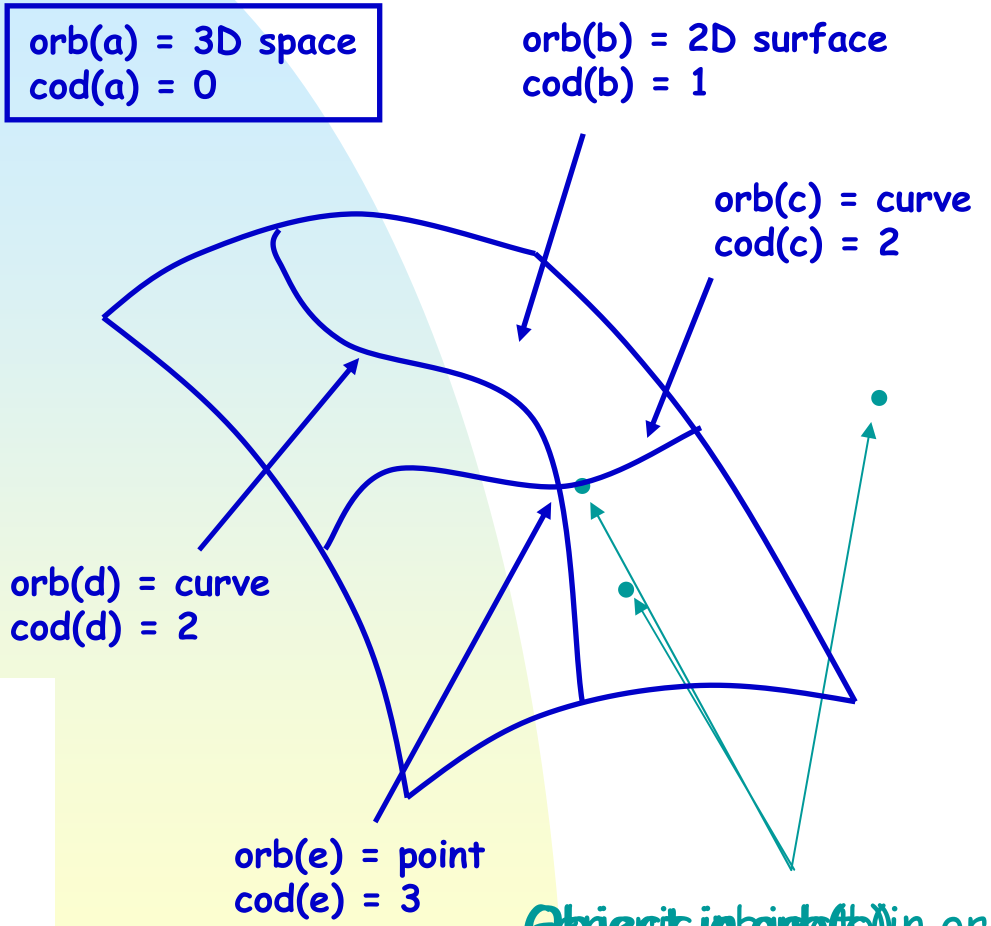
- The *closure hierarchy* of all possible Jordan structures is a *stratification.*

- The complete n^2-dim space stratified into orbits or bundles.

Orbits vis-à-vis Bundles?

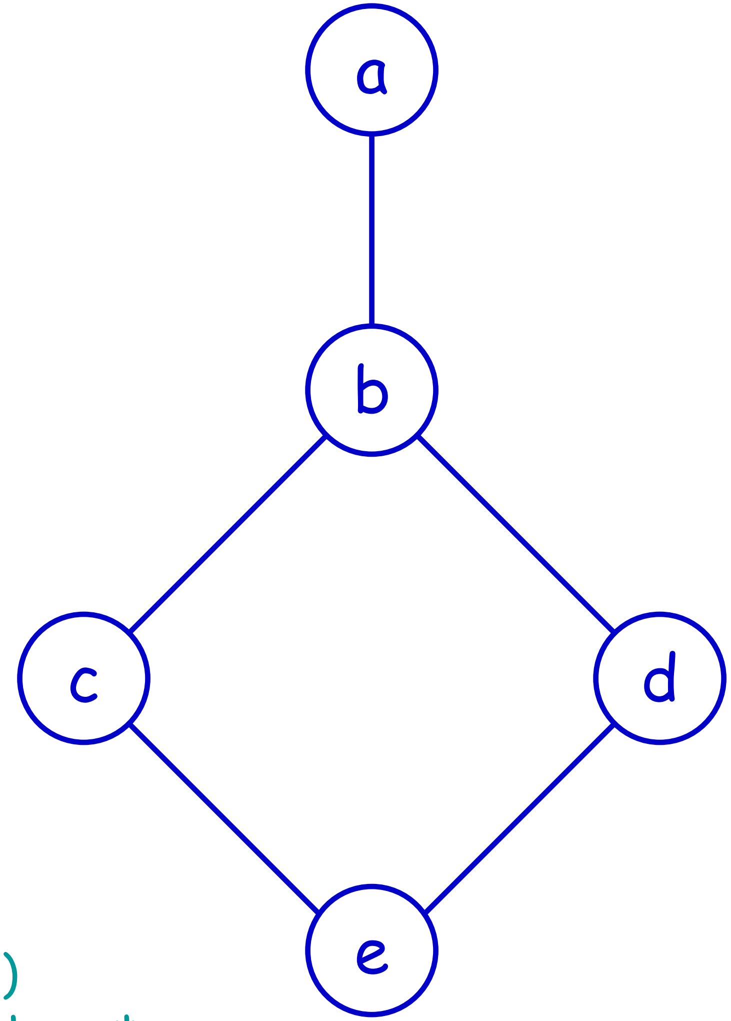
- If *A* is a limit point of orbit(B), then *A* inherits the eigenvalues of B (with algebraic multiplicities).
- If *A* is a limit point of bundle(B), then *A* either inherits all eigenvalues of B or
  - two of them coalesce (--> more degenerate)
  - one split into two different eigenvalues (--> less degenerate).

Closure hierarchy - Graph representation


orb(a) = 3D space
cod(a) = 0
orb(b) = 2D surface
cod(b) = 1
orb(c) = curve
cod(c) = 2
orb(d) = curve
cod(d) = 2
orb(e) = point
cod(e) = 3

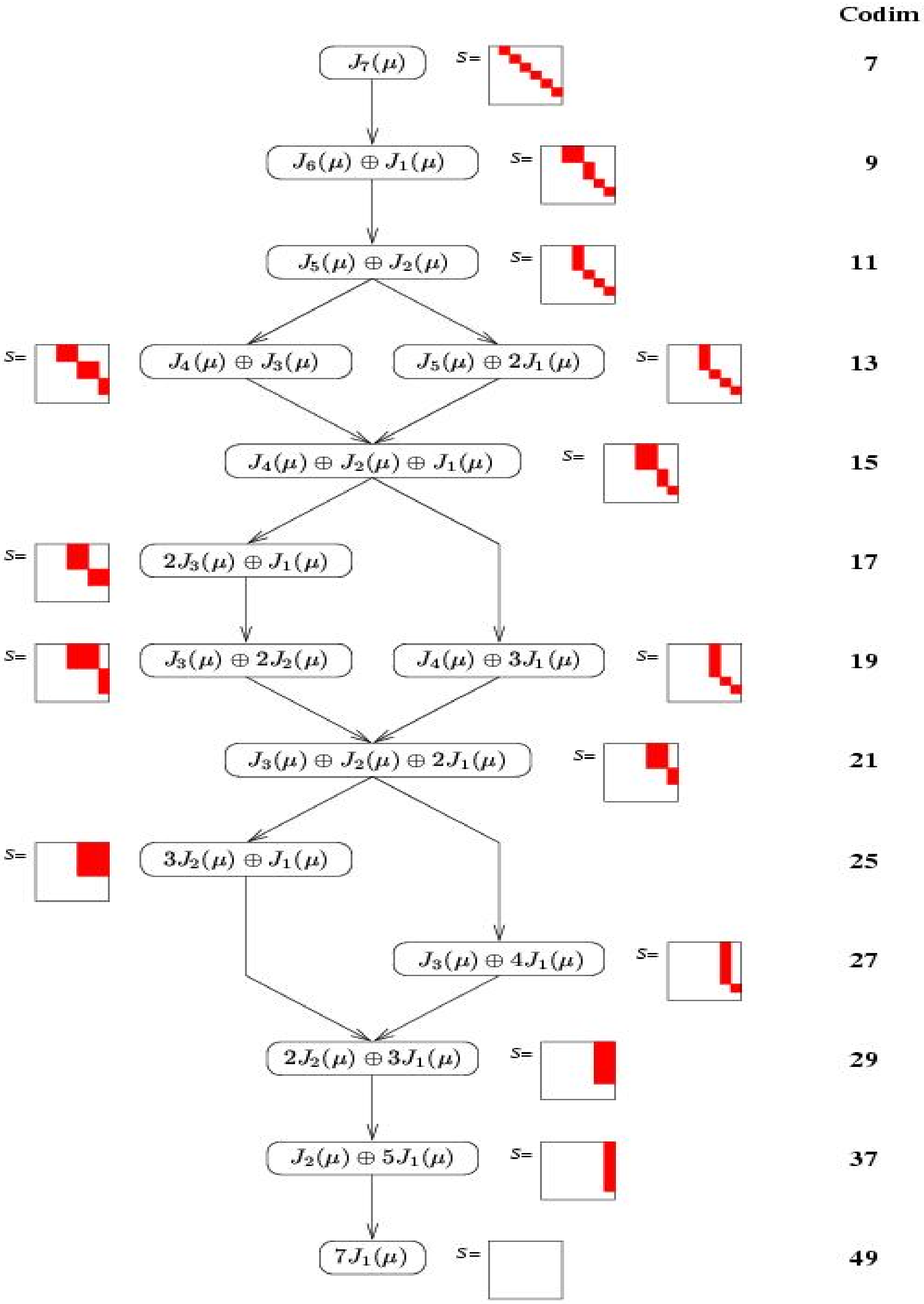
Generic object (in orb(a))
- far from any degenerate case
- close to generic cases

Object in orb(b)
- close to degenerate case
- close to generic case



Nilpotent Orbit Stratification

- Deformation of stairs and …
- versal deformations



Codim

$J_7(\mu)$ S= 7

$J_6(\mu) \oplus J_1(\mu)$ S= 9

$J_5(\mu) \oplus J_2(\mu)$ S= 11

S= $J_4(\mu) \oplus J_3(\mu)$ $J_5(\mu) \oplus 2J_1(\mu)$ S= 13

$J_4(\mu) \oplus J_2(\mu) \oplus J_1(\mu)$ S= 15

S= $2J_3(\mu) \oplus J_1(\mu)$ 17

S= $J_3(\mu) \oplus 2J_2(\mu)$ $J_4(\mu) \oplus 3J_1(\mu)$ S= 19

$J_3(\mu) \oplus J_2(\mu) \oplus 2J_1(\mu)$ S= 21

S= $3J_2(\mu) \oplus J_1(\mu)$ 25

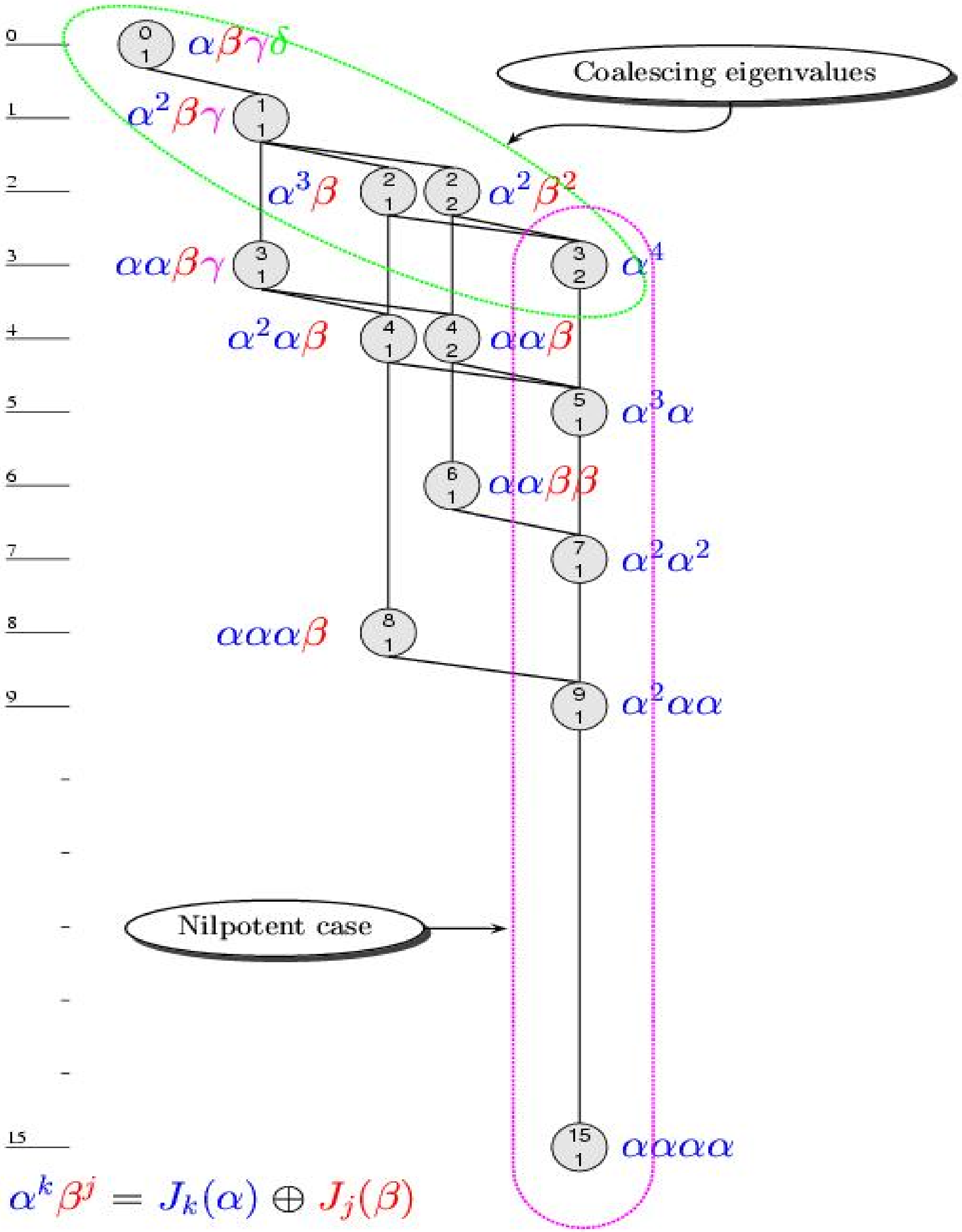
$J_3(\mu) \oplus 4J_1(\mu)$ S= 27

$2J_2(\mu) \oplus 3J_1(\mu)$ S= 29

$J_2(\mu) \oplus 5J_1(\mu)$ S= 37

$7J_1(\mu)$ S= 49

4x4 Matrix Bundle Strati-fication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15
$\alpha\beta\gamma\delta$
$\alpha^2\beta\gamma$
$\alpha^3\beta$
$\alpha^2\beta^2$
$\alpha\alpha\beta\gamma$
α^4
$\alpha^2\alpha\beta$
$\alpha\alpha\beta$
$\alpha^3\alpha$
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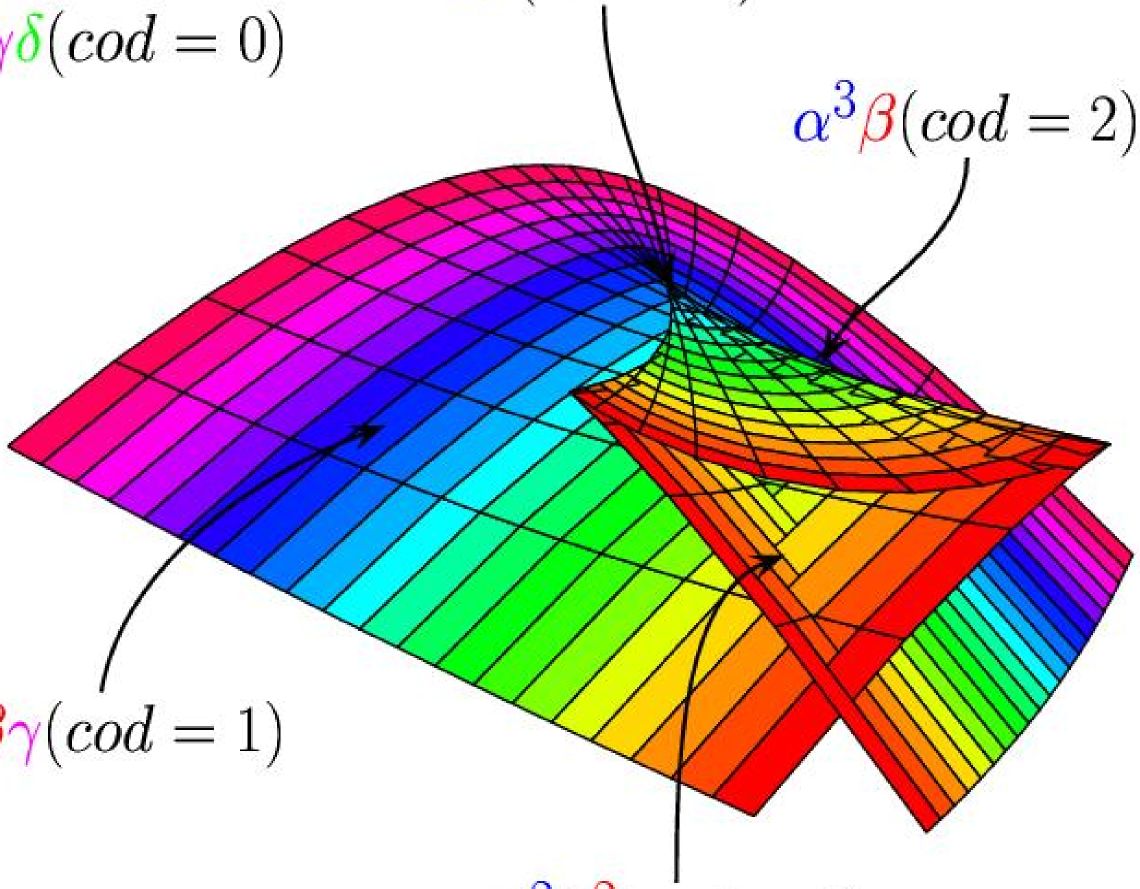
$\alpha^2\alpha^2$
$\alpha\alpha\alpha\beta$
$\alpha^2\alpha\alpha$
$\alpha\alpha\alpha\alpha$
Coalescing eigenvalues
Nilpotent case

$$\alpha^k\beta^j = J_k(\alpha) \oplus J_j(\beta)$$

Swallow Tail



$\alpha\beta\gamma\delta(cod = 0)$

$\alpha^4(cod = 3)$

$\alpha^3\beta(cod = 2)$

$\alpha^2\beta\gamma(cod = 1)$

$\alpha^2\beta^2(cod = 2)$

Kronecker Canonical Form (KCF)

Any m x n matrix pencil A - λE can be transformed into KCF:

$$P^{-1}(A-\lambda E)Q=\mathrm{diag}(L_{\varepsilon_1},\dots,L_{\varepsilon_p},J_{j_1}(\mu_1),\dots,J_{j_k}(\mu_k),N_{i_1},\dots,N_{i_p},L^T_{\eta_1},\dots,L^T_{\eta_q})$$

where

$$J_j(\mu)\equiv\begin{bmatrix}\mu-\lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \mu-\lambda\end{bmatrix} \qquad N_j\equiv\begin{bmatrix}1 & -\lambda & & \\ & \ddots & \ddots & \\ & & \ddots & -\lambda \\ & & & 1\end{bmatrix}$$

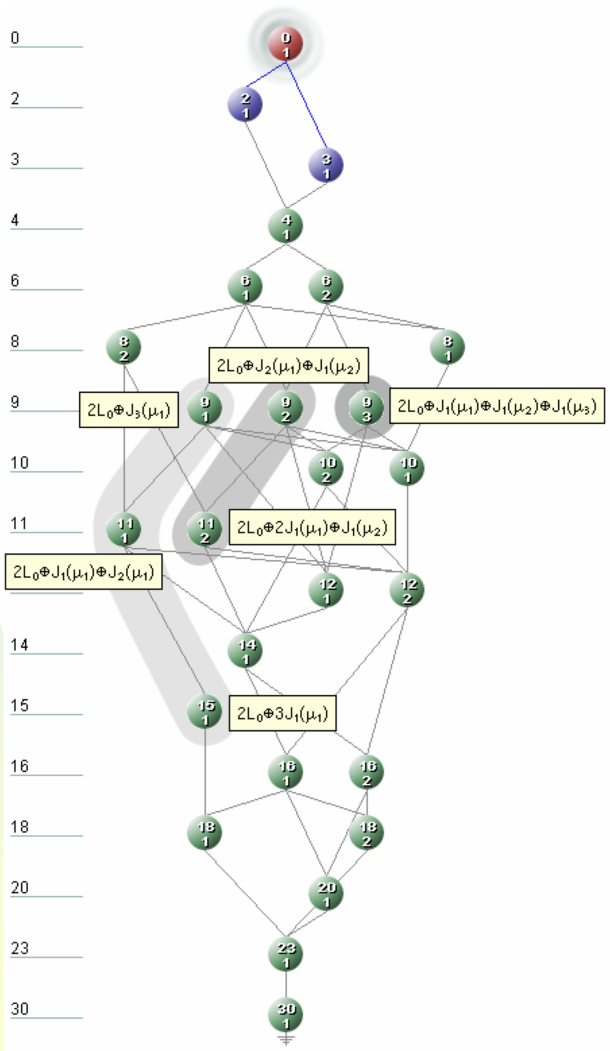
and

$$L_j\equiv\begin{bmatrix}-\lambda & 1 & & \\ & \ddots & \ddots & \\ & & -\lambda & 1\end{bmatrix} \qquad L_j^T\equiv\begin{bmatrix}-\lambda & & \\ 1 & \ddots & \\ & \ddots & -\lambda \\ & & 1\end{bmatrix}$$

3x5 Pencil Stratification

Structures with 3x3 regular part highlighted

Orbit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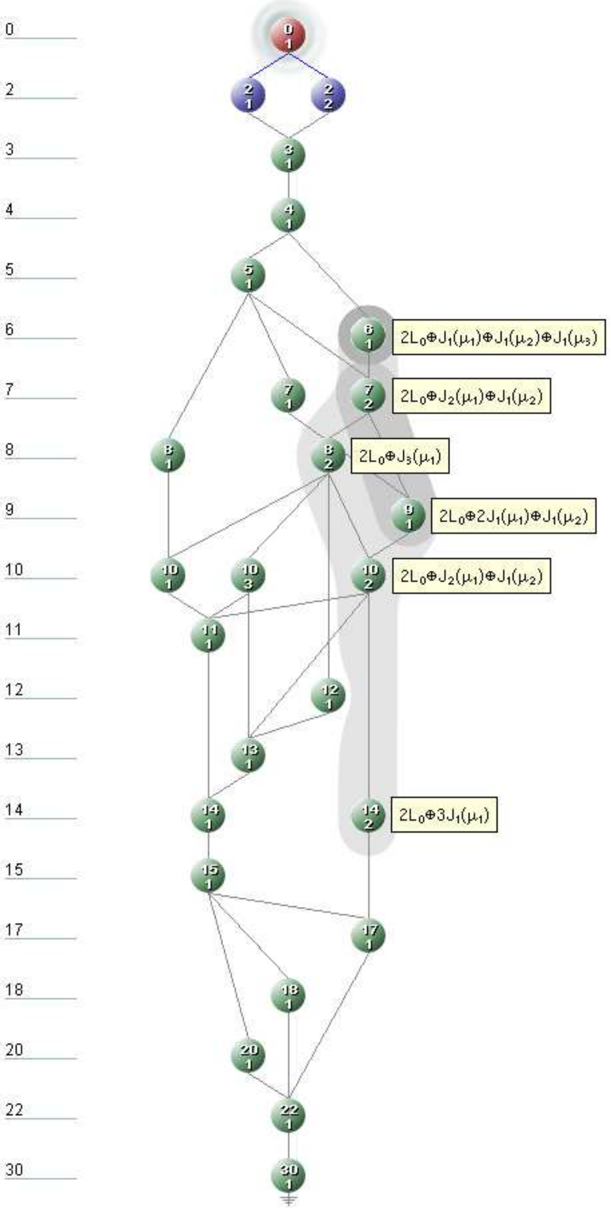
2L₀⊕J₂(μ₁)⊕J₁(μ₂)
2L₀⊕J₃(μ₁)
2L₀⊕J₁(μ₁)⊕J₁(μ₂)⊕J₁(μ₃)
2L₀⊕2J₁(μ₁)⊕J₁(μ₂)
2L₀⊕J₁(μ₁)⊕J₂(μ₁)
2L₀⊕3J₁(μ₁)

Bundle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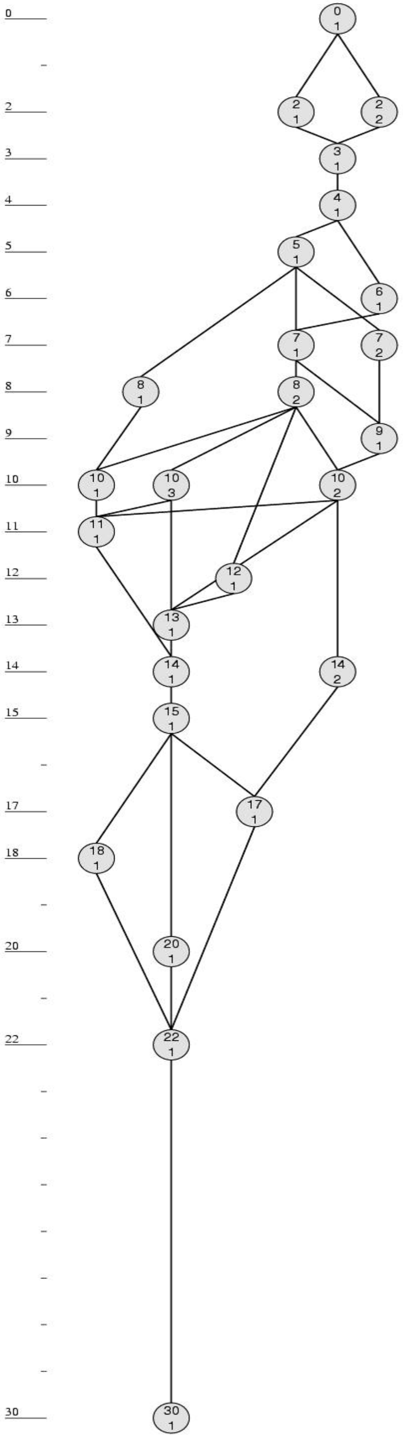
2L₀⊕J₁(μ₁)⊕J₁(μ₂)⊕J₁(μ₃)
2L₀⊕J₂(μ₁)⊕J₁(μ₂)
2L₀⊕J₃(μ₁)
2L₀⊕2J₁(μ₁)⊕J₁(μ₂)
2L₀⊕J₂(μ₁)⊕J₁(μ₂)
2L₀⊕3J₁(μ₁)

3x5 Pencil Bundle Strati-fication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10 2
11 1
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13 1
14 1
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15 1
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18 1
20 1
22 1
30 1

Nodes with no L_j^T blocks:

0:1 $L_1 \oplus L_2$
2:1 $L_0 \oplus L_3$
2:2 $2L_1 \oplus J_1(\mu_1)$
3:1 $L_0 \oplus L_2 \oplus J_1(\mu_1)$
4:1 $L_0 \oplus L_1 \oplus J_1(\mu_1) \oplus J_1(\mu_2)$
5:1 $L_0 \oplus L_1 \oplus J_2(\mu_1)$
6:1 $2L_0 \oplus J_1(\mu_1) \oplus J_1(\mu_2) \oplus J_1(\mu_3)$
7:1 $2L_0 \oplus J_2(\mu_1) \oplus J_1(\mu_2)$
7:2 $L_0 \oplus L_1 \oplus 2J_1(\mu_1)$
8:2 $2L_0 \oplus J_3(\mu_1)$
9:1 $2L_0 \oplus 2J_1(\mu_1) \oplus J_1(\mu_2)$
10:2 $2L_0 \oplus J_1(\mu_1) \oplus J_2(\mu_1)$
14:2 $2L_0 \oplus 3J_1(\mu_1)$

$$Ex'(t) = Fx(t) + Gu(t)$$

Controllability pencil:
$\mathcal{C}(E, F, G) = [G|F] - \lambda[0|E]$

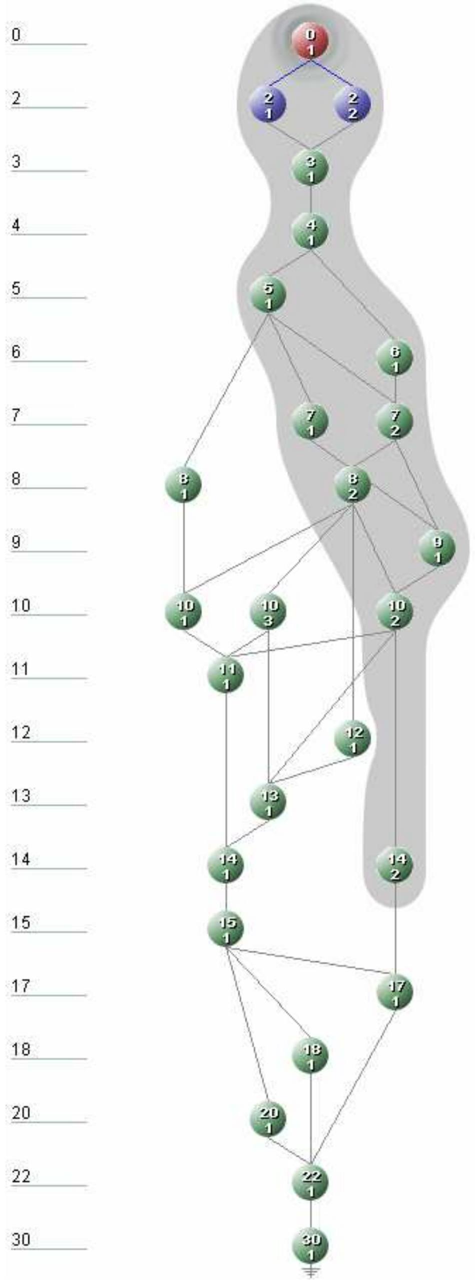
3 states and 2 inputs

3x5 Bundle stratification:

- matrix pencil

- matrix pair

A - λB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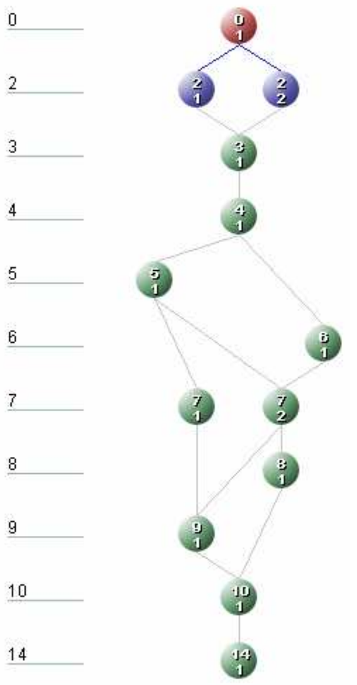
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7 2
8 1
9 1
10 1
14 1

[C,F]-λ[0,E]

The StratiGraph Tool

- The stratification theory is complex and sometimes hard to apply to real-world problems
- StratiGraph is an “API/GUI” to stratifications
- Provides qualitative & quantitative information
- Being extended to include specialized matrix pencils that directly maps into real applications
- StratiGraph is implemented in Java and interacts with Matlab (where all matrix computations are performed)

StratiGraph functionality

- Interactive presentation of stratifications
  - represented as graphs
  - complete picture or only in the close vicinity
- Upper and lower bounds to nearby structures
- Find matrices corresponding to a specified nearby structure
- Cases implemented or in progress:

$A-\lambda I$ **Matrices**

$A-\lambda E$ **Matrix pencils**

$\begin{bmatrix} A-\lambda E & B \end{bmatrix}$ **Pairs of matrices**

$\begin{bmatrix} A-\lambda E & B \\ C & 0 \end{bmatrix}$ **Triples of matrices**

$\begin{bmatrix} A-\lambda E & B \\ C & D \end{bmatrix}$ **Quadruples of matrices**

Distance to nearby structures?

- Given: $A - \lambda E$
- Find upper and lower bounds on distance to closest pencil with a specified KCF

Upper bounds

- By "imposing" the required rank and nullity conditions with a staircase regularizing algorithm, (i.e., we compute the corresponding matrices)
- The size of imposed perturbations $(\delta A, \delta E)$ gives the upper bound

Lower bounds

- Use characterization of tangent space of the orbit
- E.g., tan(A – λE) = (XA – AY) – λ(XE – EY) ∀ X, Y
  = range(T), where

$$T = \begin{bmatrix} A^T \otimes I_m & -I_m \otimes A \\ E^T \otimes I_m & -I_n \otimes E \end{bmatrix}$$

- Given c = cod(A – λE), a lower bound to a pencil with codimension c+d is

$$\left\| (\partial A, \partial E) \right\|_F \geq \frac{1}{\sqrt{m+n}} \left(\sum_{i=2mn-c-d+1}^{2mn} \sigma_i(T) \right), \text{ where } \sigma_i(T) \geq \sigma_{i+1}(T)$$

Lower bounds (cont'd)

- Similar matrix representations for the tangent space of matrices, pairs etc
  - Matrices:

$$T = I_n \otimes A - A^T \otimes I_n$$

  - Matrix pairs:

$$T = \begin{bmatrix} A \otimes I_n - I_n \otimes A^T & B \otimes I_n & 0 \\ -I_n \otimes B^T & 0 & B \otimes I_n \end{bmatrix}$$

etc.

StratiGraph Demo Tour

Session starts from Matlab

- 1st:
    - Input Jordan structure information for a matrix
    - Bundle stratification - explore structures in the hierarchy
    - Explore StratiGraph menues and functions

StratiGraph

- 2nd:
    - Enter matrix pencil $A - \lambda B$ from Matlab
    - Compute the KCF and display it in StratiGraph
    - Explore the closure hierarchy graph and bounds

StratiGraph

Some Conclusions

- Claim: it is possible to compute robust canonical structure information
- StratiGraph is the API to stratification theories for general matrices/pencils and system pencils in control applications
  - The stratification of and orbit (bundle) provides qualitative information about nearby structures
  - Quantitative information is given by bounds on distance to specified nearby structures

Acknowledgements

My co-workers on “canonical structures”:

- Axel Ruhe (my PhD supervisor)
- Jim Demmel, UC Berkeley
- Erik Elmroth (PhD’95)
- Alan Edelman, MIT
- Petter Wiberg (PhLic’98, ->Toronto)
- Pedher Johansson (PhD student)
- Stefan Johansson (PhD student)

In addition, thanks to several “great sources of inspiration” in the community!

Matrix Stratification Epilogue

- While stratigraphy is the key to understanding the geological evolution of the world, StratiGraph is the entry to understanding the "geometrical evolution" of orbits and bundles in the "world" of matrices and matrix pencils.
- But remember these worlds grow exponentially with matrix size!

- Thanks!