THE DECAY RATE OF THE SOLUTION TO A TRIDIAGONAL LINEAR SYSTEM WITH A VERY SPECIAL RIGHT HAND SIDE

CARL CHRISTIAN KJELGAARD MIKKELSEN *

1. Introduction. This is a short note which deals with a detail in the analysis of the truncated SPIKE algorithm [2], [3] for systems which are strictly diagonally dominant by rows. It contains the proof of Theorem 3.9 [1].

2. The main result. There is only one result namely the following theorem

THEOREM 2.1. *Let* $\{(a_i, b_i, c_i)\}_{i=1}^n$ *be a finite sequence, such that* $a_i \neq 0$, *and*

$$\max_{i=1,\dots,n} \frac{|b_i| + |c_i|}{|a_i|} = \epsilon < 1.$$

If the vector x *given by*

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & & \\ c_2 & \ddots & \ddots & \\ & \ddots & & b_{n-1} \\ & & c_n & a_n \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_n \end{bmatrix},$$

exhibits the smallest possible decay rate, i.e. if

$$|x_1| = \epsilon^n \tag{2.1}$$

then

$$c_i = 0, \quad \textit{and} \quad |b_i| = \epsilon |a_i|, \tag{2.2}$$

for $i = 1, 2, \dots n$.

Proof. We prove the theorem using the Thomas algorithm [4], which is designed to solve tridiagonal systems of the form

$$c_i x_{i-1} + a_i x_i + b_i x_{i+1} = f_i, \quad i = 1, 2, \dots, n,$$

where x_0, and x_{n+1} are given in advance. If the system is strictly diagonally dominant by rows with degree $d = \epsilon^{-1} > 1$ then the solution can be computed as follows

$$x_i = p_i x_{i+1} + q_i, \quad i = 1, \dots, n,$$

where the coefficient p_i and q_i are given by

$$p_0 = 0, \quad p_i = \frac{-b_i}{a_i + c_i p_{i-1}}, \quad i = 1, 2, \dots, n,$$

and

$$q_0 = x_0, \quad q_i = \frac{f_i - c_i q_{i-1}}{a_i + c_i p_{i-1}}, \quad i = 1, 2, \dots, n.$$

*Department of Computer Science, Purdue University (`cmikkels@cs.purdue.edu`)

We claim that $|p_i| \le \epsilon$, for $i = 0, 1, 2, \ldots, n$. If $b_i = 0$ then $p_i = 0$, and there is nothing to show. Assuming $|p_{i-1}| \le \epsilon < 1$, and $b_i \neq 0$, we have

$$\begin{aligned} |p_i| = \frac{|b_i|}{|a_i + c_i p_{i-1}|} &\le \frac{|b_i|}{|a_i| - |c_i|\epsilon} \\ &\le \frac{|b_i|}{\epsilon^{-1}(||b_i| + |c_i|) - |c_i|\epsilon} = \frac{|b_i|}{\epsilon^{-1}|b_i| + (\epsilon^{-1} - \epsilon)|c_i|} \le \epsilon, \quad (2.3) \end{aligned}$$

because $\epsilon \le 1$, implies $(\epsilon^{-1} - \epsilon)|c_i| \ge 0$.

In our case $x_0 = x_{n+1} = 0$, and $f_i = 0$ for $i = 1, 2, \ldots, n-1$, while $f_n = b_n$. It follows that

$$q_i = 0, \quad i = 0, 1, 2, \ldots, n-1,$$

while

$$q_n = \frac{b_n}{a_n + c_n p_{n-1}}, \quad \text{and} \quad |q_n| \le \epsilon.$$

It follows that

$$x_n = q_n, \quad x_i = \left(\Pi_{j=i}^{n-1} p_i\right) q_n,$$

which implies that

$$|x_i| \le \epsilon^{n-i+1}.$$

Now suppose $|x_1|$ assumes the largest possible value, namely

$$|x_1| = \epsilon^n$$

then we must have

$$|p_i| = \epsilon, \quad i = 1, 2, \ldots, n-1, \quad \text{and} \quad |q_n| = \epsilon.$$

Now, we claim that this can only happen if $c_i = 0$, for $i = 1, 2, \ldots, n$. From (2.3) we see that we actually have

$$\frac{|b_i|}{|a_i| - |c_i|\epsilon} = \epsilon,$$

for $i = 1, 2, \ldots, n-1$, as well as $i = n$. It follows, that

$$\epsilon^2 |c_i| = \epsilon |a_i| - |b_i|.$$

However, $\epsilon |a_i| \ge |b_i| + |c_i|$, leaving us with

$$\epsilon^2 |c_i| = \epsilon |a_i| - |b_i| \ge |c_i|,$$

from which we deduce $|c_i| = 0$, because $\epsilon < 1$. □

In short, if a tridiagonal matrix which is strictly diagonally dominant by rows, exhibits the slowest possible decay rate, then it is actually bidiagonal, and the ratio $|b_i|/|a_i|$ is fixed. In our experience the spikes always decay much faster than the worst case.

REFERENCES

[1] C. C. K. Mikkelsen, M. Manguoglu Analysis of the truncated SPIKE algorithm Submitted to SIMAX, August 2008

[2] E. Polizzi, A. Sameh *A parallel hybrid banded system solver: the SPIKE algorithm.* Parallel computing, Vol. 32 (2006), No. 2, pp 177-194.

[3] E. Polizzi, A. Sameh *SPIKE: A parallel enviroment for solving banded linear systems* Computers & Fluids, Vol. 36 (2007), pp. 113-120.

[4] J. C. Strikwerda *Finite difference schemes and partial differential equations* 1st edition, Wadsworth & Brooks/Cole, 1989.