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Contents

Chapter 1

Introduction

Let $\Omega = \mathbb{R} \times (0, \infty)$ and consider the pure initial value problem of finding a function $u \in C^\infty(\Omega)$ such that

$$u_t = u_{xx}$$

and

$$u(x,t) \to \sin(x)$$

for $(x,t) \to (x,0)$ with $t > 0$. This problem has at least one solution $u \in C^\infty(\Omega)$ given by

$$u(x,t) = e^{-t}\sin(x)$$

Consider the standard explicit finite difference method for the pure initial value problem

$$v_j^{n+1} = \frac{1}{2}v_{j+1}^n + \frac{1}{2}v_{j-1}^n$$

on the grid $\Sigma_h = \{(jh, \frac{1}{2}nh^2) \mid j \in \mathbb{Z},\ n \in \{0,1,2,\dots\}\}$ with $v_j^0 = \sin(jh)$. By induction on n we discover that

$$v_j^n = \cos(h)^n \sin(jh)$$

and two applications of l'Hospital's rule gives

$$\cos(h)^n \to e^{-t} \quad \text{as} \quad n \to \infty \quad \text{with} \quad \frac{1}{2}nh^2 = t.$$

so that

$$v_j^n \to u(x,t)$$

for $n \to \infty$ with $\frac{1}{2}nh^2 = t$, $jh = x$.

This is a thesis on the differential equation

$$u_t - au_{xx} - bu_x - cu = f$$

on various domains $\Omega \subset \mathbb{R}^2$. We treat the pure initial value problem as well as the initial-boundary value problem with Dirichlet boundary conditions.

We use finite difference methods to generate a series of grid functions which are shown to converge to a function

$$u : \Omega \to \mathbb{R}$$

which in turn is shown to satisfy the differential equation as well as the initial-boundary conditions.

This thesis is based on the works of Fritz John [1] and I.G Petrowski [2]. Fritz John has solved the pure initial value problem for the general equation, while Petrowski has shown how to solve the initial-boundary value problem for the heat equation.

In chapter 2 we develop the necessary notation for the finite difference method. We also discuss a method for generating a smooth function from a series of grid functions. Chapter 3 deals with the pure initial value problem for the parabolic equation

$$u_t - au_{xx} - bu_x - cu = f \tag{1.1}$$

We derive an existence and uniqueness theorem and establish estimates of the maximum norm of solutions and the derivatives. It is essential to understand the concept of an "invariant" finite difference scheme which we introduce in this chapter. Chapter 4 is not relevant for the following chapters but contains an analysis of the error, i.e. the difference between the exact solution and the finite difference approximation for a very simple problem. We show that it is possible to do a Taylor-like expansion of the error in the step size h with coefficients given as the solutions of a set of recursively defined differential equations. This result has great practical importance.

In chapter 5 we study the initial-boundary value problem for our differential equation on a fairly general kind of bounded domain. Assuming that the initial-boundary value problem is well-posed, i.e. admits a unique solution we develop estimates for the maximum norm of the solution. In chapter 6 we show that it is possible to estimate the maximum norm of u_x relative to the maximum norm of u on every compact set $K \subset \Omega$. The results of chapters 5 and 6 serve as a source of inspiration for chapters 7 and 8. Chapter 7 deals with the initial-boundary value problem for the heat equation with Dirichlet boundary values. We derive an existence and uniqueness theorem. Chapter 8 extends the theorems of chapter 7 to cover the initial-boundary value problem for the general equation (1.1)

Chapter 2

Notation and Preliminaries

In this chapter we develop the necessary notation for the finite difference method. We also discuss a method for generating smooth functions from a series of grid functions.

Choose $h, k > 0$ and consider the grid $\Sigma_{h,k}(\Omega)$ defined by

$$\Sigma_{h,k}(\Omega) = \{(jh, nk) \mid j, n \in \mathbb{Z}\} \cap \Omega.$$

A grid function v is a map $v : \Sigma_{h,k}(\Omega) \to \mathbb{R}$. We shall use the notation $v_j^n = v(jh, nk)$ for the function values of v.

Let $\Omega = \mathbb{R}^2$ and consider the shift operators E_x and E_t defined by

$$(E_x v)_j^n = v_{j+1}^n \qquad (E_t v)_j^n = v_j^{n+1}$$

They are clearly invertible with

$$(E_x^{-1} v)_j^n = v_{j-1}^n \qquad (E_t^{-1} v)_j^n = v_j^{n-1}$$

Also $E_x E_t = E_t E_x$ because

$$(E_x E_t v)_j^n = (E_t v)_{j+1}^n = v_{j+1}^{n+1} = (E_x v)_j^{n+1} = (E_t E_x v)_j^n,$$

and as a direct consequence

$$E_x^{-1} E_t^{-1} = (E_t E_x)^{-1} = (E_x E_t)^{-1} = E_t^{-1} E_x^{-1}.$$

The shift operators are used to define the first order difference operators $\delta_x, \delta_{\overline{x}}, \delta_t$ and $\delta_{\overline{t}}$ by

$$\delta_x = \frac{E_x - 1}{h} \qquad \delta_{\overline{x}} = \frac{1 - E_x^{-1}}{h}$$
$$\delta_t = \frac{E_t - 1}{k} \qquad \delta_{\overline{t}} = \frac{1 - E_t^{-1}}{k}$$

These are generally known as the forward space, the backward space, the forward time and the backward time operators. Clearly they are commutable as they are but linear combinations of commuting operators.

Any continuous function $u : \Omega \to \mathbb{R}$ may be regarded as a grid function through its restriction to $\Sigma_{h,k}(\Omega)$. For differentiable functions u the mean value theorem relates the first order difference quotient to the first order derivatives, as

$$(\delta_x u)(x,t) = \frac{u(x+h,t) - u(x,t)}{h} = u_x(\xi, t)$$

for some ξ between x and $x + h$. Second and higher order difference operators are defined in terms of the first operators

$$\begin{aligned}\delta_{xx} &= \delta_x \delta_x \\ \delta_{xt} &= \delta_x \delta_t = \delta_t \delta_x = \delta_{tx} \\ \delta_{tt} &= \delta_t \delta_t\end{aligned}$$

and so forth.

Of special importance to our work are the space central operators $\delta_{x\overline{x}}$ and $\mu_x = \frac{1}{2}(\delta_x + \delta_{\overline{x}})$. If the partial derivative u_x exists the mean value theorem gives

$$\begin{aligned}\mu_x(x,t) &= \frac{1}{2}\left\{\frac{u(x+h,t) - u(x,t)}{h} + \frac{u(x,t) - u(x-h,t)}{h}\right\} \\ &= \frac{u(x+h,t) - u(x-h,t)}{2h} = u_x(\xi, t)\end{aligned}$$

for some ξ in $[x - |h|, x + |h|]$.

If the second order derivative u_{xx} exists, then by Taylor's formula

$$\left[\delta_{x\overline{x}} u\right](x,t) = \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} = \frac{u_{xx}(\nu_1, t) + u_{xx}(\nu_2, t)}{2},$$

for some ν_1 between x and $x + h$ and ν_2 between x and $x - h$ and in the event of *continuous* u_{xx} we have a ν between ν_1 and ν_2, such that

$$\left[\delta_{x\overline{x}} u\right](x,t) = u_{xx}(\nu, t),$$

because the mean value $\frac{u_{xx}(\nu_1,t)+u_{xx}(\nu_2,t)}{2}$ clearly falls within the range of the restriction of u_{xx} to the interval between ν_1 and ν_2.

In general a divided difference may be related to a derivative through a Taylor-expansion as the following lemma shows

2.1 Lemma. *If $g : \mathbb{R} \to \mathbb{R}$ is m times differentiable, then*

$$[\delta_x^m g](x) = \frac{1}{m!} \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} j^m g^{(m)}(\xi_j)$$

for some ξ_j between x and $x + jh, j = 1, 2, \ldots, m$. In particular if $g^{(m)}$ is bounded we have the estimate

$$|[\delta_x^m g](x)| \leq C_m \|g^{(m)}\|_\infty$$

with $C_m = \frac{1}{m!} \sum_{j=0}^{m} \binom{m}{j} j^m$.

Proof. By definition

$$[\delta_x^m g](x) = h^{-m} \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} g(x + jh)$$

and by Taylor's theorem

$$g(x + jh) = g(x) + \sum_{k=1}^{m-1} \frac{1}{k!} g^{(k)}(x)(jh)^k + \frac{1}{m!} g^{(m)}(\xi_j)(jh)^m$$

for some ξ_j between x and $x + jh, j = 1, 2, \ldots m$. Obviously we need to show that the coefficients of $g(x)$ and $g^{(k)}, k = 1, 2, \ldots m - 1$ are zero. This reduces to showing that

$$\sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} j^k = 0, \quad k = 0, 1, 2, \ldots, m - 1.$$

or equivalently that

$$\sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} p_{m-1}(j) = 0$$

for all polynomials of degree at most $m - 1$. This is trivial if one chooses the right basis ! Let $(j)_k = j(j - 1) \ldots (j - k + 1), k = 0, 1, 2, \ldots$. Observe how the binomial theorem gives

$$(1 + x)^m = \sum_{j=0}^{m} \binom{m}{j} x^j,$$

so that $\sum_{j=0}^{m} \binom{m}{j} (-1)^j = 0$ and

$$(m)_k (1 + x)^{m-k} = \frac{d^k}{dx^k} (1 + x)^m = \sum_{j=0}^{m} \binom{m}{j} (j)_k x^{j-k}$$

and by inserting $x = -1$ we get

$$0 = \sum_{j=0}^{m} \binom{m}{j} (j)_k (-1)^{j-k}$$

for $k = 0, 1, 2, \ldots m-1$ and we are done with a single multiplication by $(-1)^{m+k-2j}$.

The following corollary is immediate but will prove useful in chapter 3.

2.2 Corollary. *If $g : \mathbb{R} \to \mathbb{R}$ is m times differentiable and if $g^{(m)}$ is uniformly continuous then*

$$[\delta_x^m g](x) - g^{(m)}(x) \to 0, \quad h \to 0$$

independently of x.

Proof. Given $\epsilon > 0$ there exists $\delta > 0$ so that

$$|x - y| < \delta \Rightarrow |g^{(m)}(x) - g^{(m)}(y)| < \epsilon.$$

Choose h so small that $mh < \delta$. Write

$$[\delta_x^m g](x) = \frac{1}{m!} \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} j^m g^{(m)}(\xi_j)$$

for some ξ_j between x and $x + jh, j = 1, 2, \dots m$. Then

$$|x - \xi_j| < jh \leq mh < \delta \quad \text{and} \quad |g^{(m)}(x) - g^{(m)}(\xi_j)| < \epsilon$$

so that

$$|[\delta_x^m g](x) - g^{(m)}(x)| \leq C_m \epsilon$$

independently of x.

We now turn to the question of generating smooth functions from a series of grid functions.

We need the following lemma which I believe is due to Cantor.

2.3 Lemma. *Let A be a countable set and let $f_n : A \to \mathbb{R}, n \in \{1, 2, 3, \dots\}$ be a sequence of functions. If $\{f_n(p)\}_{n=1}^{\infty}$ is bounded for every point $p \in A$, then there exists a subsequence, i.e. a strictly increasing function $g : \mathbb{N} \to \mathbb{N}$ such that $\{f_{g(n)}(p)\}_{n=1}^{\infty}$ is convergent for all $p \in A$.*

Proof. Let $A = \{p_1, p_2, p_3, \dots\}$. The sequence $\{f_n(p_1)\}_{n=1}^{\infty}$ is bounded and we may therefore extract a subsequence $\{f_{1n}\}_{n=1}^{\infty}$ such that $\{f_{1n}(p_1)\}_{n=1}^{\infty}$ is convergent. The sequence $\{f_{1n}(p_2)\}_{n=1}^{\infty}$ is bounded and we may therefore extract a subsequence $\{f_{2n}\}_{n=1}^{\infty}$ from $\{f_{1n}\}_{n=1}^{\infty}$ such that $\{f_{2n}(p_2)\}_{n=1}^{\infty}$ is convergent. In general we extract from $\{f_{jn}\}_{n=1}^{\infty}$ a subsequence $\{f_{j+1,n}\}_{n=1}^{\infty}$ such that $\{f_{j+1,n}(p_{j+1})\}_{n=1}^{\infty}$ is convergent. Now consider the subsequence $\{f_{nn}\}_{n=1}^{\infty}$. The tail $\{f_{nn}\}_{n=j}^{\infty}$ is merely a subsequence of $\{f_{jn}\}_{n=j}^{\infty}$ and thus we discover that $\{f_{nn}(p_j)\}_{n=1}^{\infty}$ is convergent for each $p_j \in A$.

In chapter 7 we shall need the following corollary.

2.4 Corollary. *Let $A_1 \subset A_2 \subset A_3 \subset \dots$ be an increasing sequence of sets such that $A = \cup_{n=1}^{\infty} A_n$ is countable and let $f_n : A_n \to \mathbb{R}, n \in \mathbb{N}$ be a sequence of functions. If for all m and for all $p \in A_m$ the sets*

$$\{f_n(p) \,|\, n \geq m\}$$

are bounded then there exists a subsequence $g : \mathbb{N} \to \mathbb{N}$ with the following property

$$\forall\, p \in A \,\exists\, N_p \,:\, \{f_{g(n)}(p)\}_{n=N_p}^{\infty} \text{is defined and convergent.}$$

Proof. Let $p \in A$. Let m be the smallest integer such that $p \in A_m$ for some value of m. Then $p \in A_n$ for all $n \geq m$ and the sequence $\{f_n(p)\}_{n=m}^{\infty}$ is defined and bounded by assumption. We now extend f_n artificially to the entire set A by the following definition

$$\hat{f}_n(p) = \begin{cases} f_n(p) & \text{if} \quad p \in A_n \\ 0 & \text{if} \quad p \notin A_n \end{cases}$$

Then $\{\hat{f}_n(p)\}_{n=0}^{\infty}$ is defined and bounded for all p and by the previous lemma there exists a subsequence $g : \mathbb{N} \to \mathbb{N}$ such that $\{\hat{f}_{g(n)}(p)\}_{n=0}^{\infty}$ is convergent for every $p \in A$. Pick $p \in A$ at random. Then $p \in A_m$ for some m. Since g is strictly increasing there exists an N such that $g(n) \geq m$ for all $n \geq N$. Pick $n \geq N$. Then $p \in A_{g(n)}$ because $p \in A_m \subset A_{g(n)}$ and $\hat{f}_{g(n)}(p) = f_{g(n)}(p)$. Thus $\{f_{g(n)}(p)\}_{n=N}^{\infty}$ is defined and convergent. This completes the proof.

2.5 Remark. Thus by passing to a subsequence we may define a function $f : A \to \mathbb{R}$ through the relation

$$f(p) = \lim_{n\to\infty} f_n(p)$$

for all $p \in A$.

Now pick $h_0 > 0, \lambda > 0$ and set $h_\nu = 2^{-\nu} h_0, k_\nu = \lambda h_\nu^2$ and consider the grids

$$\Sigma_\nu = \Sigma_{h_\nu,k_\nu}(\mathbb{R}^2), \quad \nu = 0, 1, 2, \ldots$$

Let $\Sigma = \cup_{\nu=0}^{\infty} \Sigma_\nu$. Σ is countable because it is a countable union of countable sets. Σ is dense, because the distance between any given point and the closest point in Σ_ν is at most $\frac{1}{2}\sqrt{h_\nu^2 + k_\nu^2}$ which tends to zero as ν tends to infinity.

Let there be given a sequence $\{w^\nu\}_{\nu=1}^{\infty}$ of grid functions $w^\nu : \Sigma_\nu \to \mathbb{R}$.

2.6 Lemma. *If the grid functions are bounded uniformly and independently of the grid,*

$$\exists M \ \forall \nu \ \forall (x,t) \in \Sigma_\nu : |w^\nu(x,t)| \leq M,$$

then there exists a subsequence $\{w^{\nu'}\}$ *such that* $\{w^{\nu'}(x,t)\}$ *converges for all* $(x,t) \in \Sigma$ *and we may define a function* $u : \Sigma \to \mathbb{R}$ *through the relation*

$$u(x,t) = \lim_{\nu'\to\infty} w^{\nu'}(x,t).$$

The function u *is uniformly bounded by* M.

Proof. Let $(x,t) \in \Sigma$ be given. Then $(x,t) \in \Sigma_\mu$ for some μ, and therefore for all $\nu \geq \mu$, because the grids are formed by *subdivision*. The sequence $\{w^\nu(x,t)\}_{\nu=\mu}^{\infty}$ is bounded by M and by corollary 2.4 we can find a convergent subsequence $\{w^{\nu'}\}$ of $\{w^\nu\}$ such that $\{w^{\nu'}(x,t)\}$ converges for all (x,t) in the *countable* set Σ.

2.7 Lemma. *If further the first order divided differences are bounded uniformly and independently of the grid,*

$$\exists L\ \forall\nu\ \forall(x,t)\ \in\ \Sigma_\nu\ :\ |\delta_x w^\nu(x,t)|, |\delta_t w^\nu(x,t)| \leq L,$$

then the function $u : \Sigma \to \mathbb{R}$ is Lipschitz continuous with constant $\sqrt{2}L$ and as such admits a unique extension to the entire domain.

Proof. Let $(x_1, t_1), (x_2, t_2)$ be a pair of distinct points in Σ. We may assume that $x_1 \leq x_2$. Then $(x_1, t_1), (x_2, t_2) \in \Sigma_\mu$ for some value of μ and

$$\begin{aligned} x_2 &= x_1 + mh_\mu \\ t_2 &= t_1 + nk_\mu, \end{aligned}$$

for some integer values of m, n. We shall omit the subscript μ for the sake of notational simplicity throughout the remainder of this proof. Now trivially

$$w(x+h,t) - w(x,t) = \frac{w(x+h,t) - w(x,t)}{h} h = [\delta_x w](x,t)h,$$

so that $|w(x_1 + h, t) - w(x_1, t)| \leq Lh$ and similarly

$$\begin{aligned} |w(x+mh,t) - w(x,t)| &= \left| \sum_{j=0}^{m-1} w(x+(j+1)h,t) - w(x+jh,t) \right| \\ &= \left| \sum_{j=0}^{m-1} [\delta_x w](x+jh,t)h \right| \leq \sum_{j=0}^{m-1} |[\delta_x w](x+jh,t)h| \\ &\leq \sum_{j=0}^{m-1} Lh = mhL. \end{aligned}$$

Therefore

$$|w(x_2,t_2) - w(x_1,t_2)| = |w(x_1+mh,t_2) - w(x_1,t_2)| \leq mhL = |x_2 - x_1|L,$$

and similarly

$$|w(x_1,t_2) - w(x_1,t_1)| = |w(x_1,t_2+nk) - w(x_1,t_1)| \leq |t_2 - t_1|L,$$

so that we may finally use the Cauchy-Schwartz inequality to conclude that

$$\begin{aligned} |w(x_2,t_2) - w(x_1,t_1)| &\leq |w(x_2,t_2) - w(x_1,t_2)| + |w(x_1,t_2) - w(x_1,t_1)| \\ &\leq |x_2 - x_1|L + |t_2 - t_1|L \\ &\leq \sqrt{2}L\sqrt{(x_2-x_1)^2 + (t_2-t_1)^2}. \end{aligned}$$

The extension of u from the *dense* set Σ is trivial, but relies on the completeness of $\mathbb{R}$. The extension is also Lipschitz continuous with the same constant.

2.8 Lemma. *If further the second order divided differences $\delta_{xx}w^\nu$ are bounded uniformly and independently of the grid,*

$$\exists K\ \forall\nu\ \forall(x,t)\ \in\ \Sigma_\nu\ :\ |\delta_{xx}w^\nu(x,t)|\ \leq\ K,$$

then we have the important inequality

$$\left|\frac{w^\nu(x_2,t)-w^\nu(x_1,t)}{x_2-x_1}-[\delta_x w^\nu](x_1,t)\right|\leq K|x_2-x_1|$$

for all pairs of distinct points $(x_1,t),(x_2,t)\in\Sigma$.

2.9 Remark. The proof hinges on the successful factorisation of the operator $E^m-1-m(E-1)$, where $E=E_x$. Consider the special case of $m=5$. Our aim is to calculate $(E^5-1-5(E-1)$ in terms of $(E-1)^2$

$$\begin{aligned}E^5-1-5(E-1)&=\big(E^4+E^3+E^2+E+1-5\big)\,(E-1)\\&=\big(\{E^4-1\}+\{E^3-1\}+\{E^2-1\}+\{E-1\}\big)\,(E-1)\\&=\big(\{E^3+E^2+E+1\}+\{E^2+E+1\}+\{E+1\}+1\big)\,(E-1)^2\\&=\big(E^3+2E^2+3E+4\big)\,(E-1)^2\end{aligned}$$

Proof. As in the previous proof we write

$$x_2=x_1+mh_\nu$$

for an integer value of m and a suitable large value of ν. We omit the subscript of ν throughout the remainder of the proof. Then

$$\begin{aligned}&\frac{w(x_2,t)-w(x_1,t)}{x_2-x_1}-[\delta_x w](x_1,t)\\&\qquad=\frac{w(x_1+mh,t)-w(x_1,t)}{mh}-\frac{w(x_1+h,t)-w(x_1,t)}{h}\\&\qquad=\frac{[E_x^m w](x_1,t)-w(x_1,t)}{mh}-\frac{[E_x w](x_1,t)-w(x_1,t)}{h}\\&\qquad=\left[\frac{E_x^m-1}{mh}-\frac{E_x-1}{h}\right]w(x_1,t)\end{aligned}$$

We seek to involve the second order divided difference and must therefore factor out the operator $(E_x-1)^2$. We drop the subscript of x in E_x.

$$\begin{aligned}E^m-1-m(E-1)&=(E-1)(\overbrace{E^{m-1}+E^{m-2}+\cdots+E+1}^{\text{there are m terms}}-m)\\&=(E-1)\left\{\left(\sum_{j=0}^{m-1}E^j\right)-m\right\}\\&=(E-1)\sum_{j=0}^{m-1}\big(E^j-1\big)=(E-1)\sum_{j=1}^{m-1}\big(E^j-1\big)\end{aligned}$$

The terms $E^j - 1$ are themselves rewritten as sums, $E^j - 1 = (E-1)\sum_{q=0}^{j-1} E^q$, such that

$$\begin{aligned}\sum_{j=1}^{m-1} \left(E^j - 1\right) &= \sum_{j=1}^{m-1} (E-1) \sum_{q=0}^{j-1} E^q \\ &= (E-1) \sum_{q=0}^{m-2} \sum_{j=q+1}^{m-1} E^q = (E-1) \sum_{q=0}^{m-2} (m-1-q) E^q\end{aligned}$$

From which we may finally conclude

$$\begin{aligned}\frac{E^m - 1}{mh} - \frac{E-1}{h} = \frac{E^m - 1 - m(E-1)}{mh} &= \frac{(E-1)^2}{mh} \sum_{q=0}^{m-2} (m-1-q) E^q \\ &= \frac{h}{m} \left\{ \sum_{q=0}^{m-2} (m-1-q) E^q \right\} \left(\frac{E-1}{h} \right)^2\end{aligned}$$

so that

$$\begin{aligned}\left[\frac{E^m - 1}{mh} - \frac{E-1}{h} \right] w(x_1, t) &= \frac{h}{m} \left[\delta_{xx} \sum_{q=0}^{m-2} (m-1-q) E^q w \right] (x_1, t) \\ &= \frac{h}{m} \sum_{q=0}^{m-2} (m-1-q) [\delta_{xx} w](x_1 + qh, t),\end{aligned}$$

allowing us the pleasure of estimating

$$\begin{aligned}&\left| \left[\frac{E^m - 1}{mh} - \frac{E-1}{h} \right] w(x_1, t) \right| \\ &\qquad \le \frac{h}{m} \underbrace{\sum_{q=0}^{m-2} (m-1-q)}_{\text{sum of 1,2, ..., (m-1)}} K = \frac{h}{m} \frac{(m-1)m}{2} K = \frac{(m-1)h}{2} K \\ &\qquad \le \frac{1}{2} K |x_2 - x_1|\end{aligned}$$

Now imagine that all the conditions of lemma 2.6, 2.7 and 2.8 are met. Then the first order divided differences $\delta_x w^\nu$ are bounded uniformly and independently of the grid and by lemma 2.6 we may refine our subsequence further such that not only is $\{w^{\nu'}(x,t)\}$ still convergent for every $(x,t) \in \Sigma$ but so is $\{[\delta_x w^{\nu'}](x,t)\}$ and we may define a function $u' : \Sigma \to \mathbb{R}$ by

$$u'(x,t) = \lim_{\nu' \to \infty} [\delta_x w^{\nu'}](x,t).$$

By letting ν' tend to infinity in the inequality of lemma 2.8 we arrive at

$$\left| \frac{u(x_2,t) - u(x_1,t)}{x_2 - x_1} - u'(x_1,t) \right| \le K|x_2 - x_1|, \tag{2.1}$$

for all pairs of distinct points $(x_i, t) \in \Sigma$, $i = 1, 2$.

2.10 Lemma. *If further the second order divided differences*

$$\delta_{tx} w^{\nu}$$

are bounded uniformly and independently of the grid, then the function u' is Lipschitz continuous on Σ and admits a unique Lipschitz continuous extension to the entire domain. In addition u is everywhere differentiable with respect to x and the partial derivative is u', i.e.

$$\frac{u(x+h,t) - u(x,t)}{h} \to u'(x,t), \quad h \to 0, \quad h \in R,$$

for all (x,t).

Proof. Since the *first* order divided differences $\delta_x(\delta_x w^{\nu}), \delta_t(\delta_x w^{\nu})$ of $\delta_x w^{\nu}$ are bounded uniformly and independently of the grid we may apply lemma 2.7 to $\delta_x w$. u' may therefore be extended in the desired fashion. Now given two distinct points (x_1, t) and (x_2, t) in $\mathbb{R}^2$, we pick sequences $\{(x_{in}, t_n)\} \subset \Sigma$, such that $(x_{in}, t_n) \to (x_i, t)$ as $n \to \infty, i = 1, 2$. Applying inequality (2.1) and passing to the limit allows us to conclude

$$\left| \frac{u(x_2,t) - u(x_1,t)}{x_2 - x_1} - u'(x_1,t) \right| \leq K|x_2 - x_1|$$

The proof is finally completed by letting x_2 tend to x_1.

2.11 Remark. u will be continuously differentiable with respect to t provided that $\delta_{tt} w^{\nu}$ is bounded. We have in fact already assumed that $\delta_x \delta_t w^{\nu} = \delta_t \delta_x w^{\nu}$ is bounded. In general we require that all the divided differences of order at most $p+1$ are bounded in order to ensure that $u \in C^p$. The derivatives will be bounded uniformly and they will be Lipschitz continuous.

2.12 Remark. Lemmata 2.6, 2.7, 2.8 and 2.10 have been extracted from the text of Fritz John's book on partial differential equations [1].

Chapter 3

The Pure Initial Value Problem

In this chapter we consider the equation

$$u_t = au_{xx} + bu_x + cu + f \tag{3.1}$$

for (x, t) in the strip

$$S_T = \mathbb{R} \times [0, T]$$

and $u(x, 0) = g(x)$ given. We deal with functions a, b, c and f which are smooth with bounded derivatives. We assume that g is smooth with bounded derivatives and we assume that

$$\inf_{S_T} a > 0.$$

3.1 Definition. *A classical solution of equation* (3.1) *is a function*

$$u : S_T \to \mathbb{R}$$

such that u is differentiable, u_x is continuous and differentiable with respect to x, and u_{xx} is continuous and at each point $(x, t) \in S_T$

$$u_t(x, t) = a(x, t)u_{xx}(x, t) + b(x, t)u_x(x, t) + c(x, t)u(x, t) + f(x, t)$$

and $u(x, 0) = g(x)$.

3.2 Remark. Traditionally differentiability is defined in terms of *open* sets. Let $\Omega \subset \mathbb{R}^2$ be a open set. Let $u : \Omega \to \mathbb{R}$ be a function. We say that $u \in C^p(\overline{\Omega})$ iff $u \in C^p(\Omega) \cap C(\overline{\Omega})$ and all the derivatives of u extend continuously to the boundary of Ω. Please note that if $u \in C^1(S_T)$ then the restriction of u to the final line of $t = T$ is differentiable with respect to x for all $x \in \mathbb{R}$ and the restriction of u to a line $x = x_0$ is differentiable with respect to t for all $t \in [0, T]$.

I shall omit the word "classical" in all future references. We shall prove the following main results.

3.3 Theorem. *(Existence of solution) If* $\inf a > 0$ *and if* a, b, c, *and* f *are bounded and four times differentiable with bounded derivatives on the strip* S_T *and if* g *is four times differentiable with bounded derivatives then there exists a solution of* (3.1) *such that* u, u_x, u_t *and* u_{xx} *are bounded and Lipschitz continuous on* S_T.

Solutions are essentially bounded in terms of their initial values and the driving force f :

3.4 Theorem. *(Boundedness) If* u *solves equation* (3.1) *with* u, u_t, u_x, u_{xx} *continuous and* u, u_x *bounded then*

$$|u(x,t)| \leq \left(\sup_{\mathbb{R}} |g| + T \sup_{S_T} |f| \right) e^{CT} \quad \textit{for all} \quad (x,t) \in S_T$$

where

$$C = \max\{0, \sup_{S_T} c\}$$

This theorem clearly implies the following uniqueness theorem.

3.5 Theorem. *(Uniqueness) If* u *and* v *solve equation* (3.1) *with* u, u_x, u_t, u_{xx} *continuous and* u, u_x *bounded and similarly for* v *then* $u \equiv v$.

We shall prove these theorems using the finite difference method. We begin by considering the special case of the heat equation :

$$u_t = u_{xx}, \quad (x,t) \in \overline{\Omega} = \mathbb{R} \times [0, \infty)$$

with $u(x,0) = g(x)$ given. We choose $h, k > 0$ and introduce a grid $\Sigma_{h,k}$

$$\Sigma_{h,k} = \{(jh, nk) \mid j \in \mathbb{Z},\ n = 0, 1, 2, \dots\}$$

and replace the partial derivatives with divided differences

$$\frac{v_j^{n+1} - v_j^n}{k} = \frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{h^2}, \quad j \in \mathbb{Z}, \quad n = 0, 1, 2, \dots \tag{3.2}$$

and $v_j^0 = g_j = g(jh),\ j \in \mathbb{Z}$.

This equation is clearly equivalent to

$$v_j^{n+1} = \lambda v_{j+1}^n + (1 - 2\lambda) v_j^n + \lambda v_{j-1}^n, \tag{3.3}$$

with $\lambda = k/h^2$. Using the triangle inequality we find

$$|v_j^{n+1}| \leq \lambda |v_{j+1}^n| + |(1-2\lambda)|\, |v_j^n| + \lambda |v_{j-1}^n| \leq (2\lambda + |(1-2\lambda)|)\, \|v^n\|,$$

where

$$\|v^n\| = \sup_{j\in\mathbb{Z}} |v_j^n|.$$

In the special case of $\lambda \le \frac{1}{2}$, $|(1-2\lambda)| = 1-2\lambda$ and therefore

$$\|v^{n+1}\| \le \|v^n\|.$$

By induction on n we have the global estimate

$$|v_j^n| \le \|v^n\| \le \|v^0\| = \|g\| \le \|g\|_\infty.$$

In summary: For $\lambda \le \frac{1}{2}$ the solution to the finite difference equation is bounded uniformly in terms of the initial conditions and the bound is independent of the actual grid. Obviously this estimate implies that there is at most one solution of (3.2) and the equivalent equation (3.3) shows how to generate it from the initial condition.

Inspired by our work in chapter 2 we pick $\lambda \le \frac{1}{2}$, choose $h_0 > 0$ at random and set $h_\nu = 2^{-\nu}h_0, k_\nu = \lambda h_\nu^2$. Define the grids Σ_ν by

$$\Sigma_\nu = \Sigma_{h_\nu,k_\nu} = \{(jh_\nu, nk_\nu) \mid j \in \mathbb{Z},\ n = 0,1,2,\dots\}$$

and let $\Sigma = \cup_{\nu=0}^\infty \Sigma_\nu$ be their union. Let $v^\nu : \Sigma_\nu \to \mathbb{R}$ be *the* solution of our finite difference equation on Σ_ν. Now in order to generate a differentiable function u we must bound

$$\delta_x v^\nu, \delta_t v^\nu, \delta_{xx} v^\nu, \delta_{xt} v^\nu, \delta_{tt} v^\nu$$

and in addition we must bound

$$\delta_{xxx} v^\nu, \delta_{xtx} v^\nu$$

in order to make u_x differentiable with respect to x.

We make the important observation that if v is a solution to equation (3.2) then so is $\delta_x v$ and $\delta_t v$. The various divided differences are therefore bounded in terms of their initial values and it remains to relate those to the derivatives of g. The problem is simplified somewhat by observing, that

$$\delta_t v = \delta_{x\overline{x}} v = \delta_x \delta_{\overline{x}} v = \delta_x E_x^{-1} \delta_x v = E_x^{-1} \delta_{xx} v$$

so that we merely need to handle

$$\delta_x^m v^\nu,$$

for $m = 1,2,3,4$.

In chapter 2 we proved the following basic result.

3.6 Lemma. *If $g : \mathbb{R} \to \mathbb{R}$ is m times differentiable and if $g^{(m)}$ is bounded then*

$$\|\delta_x^m g\| \le C_m \|g^{(m)}\|_\infty$$

with $C_m = \frac{1}{m!}\sum_{j=0}^m \binom{m}{j} j^m$. In particular $C_1 = C_2 = 1, C_3 = 9$ and $C_4 = \frac{85}{3}$.

In summary: All the required divided differences will be bounded uniformly and independently of the grid provided that the initial condition g is 4 times differentiable with bounded derivatives.

As in chapter 2 we may now assume that our sequence $\{v^\nu\}$ is such that $\{Av^\nu(x,t)\}$ is convergent for all $(x,t) \in \Sigma$, where A is any one of the operators

$$Id, \delta_x, \delta_t, \delta_{xx}, \delta_{xt}, \delta_{tt}, \delta_{xxx}, \delta_{txx}$$

We may therefore define functions $u, u', u'', \dot{u} : \Sigma \to \mathbb{R}$ through

$$u(x,t) = \lim_{\nu\to\infty} v^\nu(x,t) \qquad u'(x,t) = \lim_{\nu\to\infty} [\delta_x v^\nu](x,t)$$
$$u''(x,t) = \lim_{\nu\to\infty} [\delta_{xx} v^\nu](x,t) \qquad \dot{u}(x,t) = \lim_{\nu\to\infty} [\delta_t v^\nu](x,t)$$

These functions are all densely defined, bounded and Lipschitz continuous functions and as such admit unique extensions to the entire domain of $\overline{\Omega}$. As proven in chapter 2 the function u is differentiable for $t \geq 0$ with $u_x = u', u_t = \dot{u}$. The function u' is differentiable with respect to x and the partial derivative is $u_{xx} = u'_x = u''$. Obviously u'' is closely related to $\dot{u}$ and we claim that they are in fact equal, so that $u_t = u_{xx}$ and u solves the heat equation. Observe how

$$\begin{aligned}
\delta_{xx} v^\nu(x,t) &= E_x \delta_{x\overline{x}} v^\nu(x,t) = E_x \delta_t v^\nu(x,t) = \delta_t v^\nu(x+h_\nu, t) \\
&= \delta_t v^\nu(x,t) + \frac{\delta_t v^\nu(x+h_\nu,t) - \delta_t v^\nu(x,t)}{h_\nu} h_\nu \\
&= \delta_t v^\nu(x,t) + \delta_{xt} v^\nu(x,t) h_\nu \\
&\to \dot{u}(x,t),
\end{aligned}$$

for all $(x,t) \in \Sigma$, because $\delta_{xt} v^\nu(x,t)$ is bounded independently of ν and h_ν tends to zero. Therefore $\dot{u}$ agrees with u'' on Σ and by continuity they are identical on $\overline{\Omega}$. Also by construction u agrees with g on a dense subset of $\mathbb{R}$, so by continuity u satisfies the initial condition.

We have finally proven the following theorem.

3.7 Theorem. *If the initial condition $g : \mathbb{R} \to \mathbb{R}$ is bounded and 4 times differentiable and if the derivatives $g^{(i)}, i = 1, 2, 3, 4$ are bounded, then there exists at least one function $u : \overline{\Omega} \to \mathbb{R}$, such that $u(x,0) = g(x)$, u is differentiable for $(x,t) \in \overline{\Omega}$, u_x is differentiable with respect to x for $(x,t) \in \overline{\Omega}$ and $u_t = u_{xx}$ for $(x,t) \in \overline{\Omega}$. The functions u, u_x, u_t and u_{xx} are all bounded and uniformly Lipschitz continuous on $\overline{\Omega}$.*

Recall that we picked $\lambda \leq \frac{1}{2}$ and chose $h_0 > 0$ at random ! Does our solution depend on these parameters in some subtle way ? The uniqueness theorem 3.5 will imply that our solution is independent of λ and h_0.

We now turn to the general case of equation (3.1)

$$u_t = au_{xx} + bu_x + cu + f, \quad (x,t) \in S_T$$

with $u(x,0)=g(x)$ given.

Pick $N \in \{1,2,3,\dots\}$ at random, set $k=T/N$ and pick $h>0$ at random and consider the grid $\Sigma_{h,k}$ defined by

$$\Sigma_{h,k} = \{(jh,nk) \mid j \in \mathbb{Z},\ n=0,1,2,\dots,N\}$$

and replace the partial derivatives with divided differences

$$\frac{v_j^{n+1}-v_j^n}{k} = a_j^n \frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{h^2} + b_j^n \frac{v_{j+1}^n - v_{j-1}^n}{2h} + c_j^n v_j^n + f_j^n \tag{3.4}$$

with $j \in \mathbb{Z}$, $n=0,1,2,\dots,N-1$ and $v_j^0 = g_j = g(jh)$, $j \in \mathbb{Z}$. This is the standard explicit scheme for our differential equation. It is equivalent to the iteration

$$\begin{aligned} v_j^{n+1} = \left(a_j^n\lambda + \frac{1}{2}b_j^n\lambda h\right) v_{j+1}^n + \left(1 - 2a_j^n\lambda + c_j^n\lambda h^2\right) v_j^n \\ + \left(a_j^n\lambda - \frac{1}{2}b_j^n\lambda h\right) v_{j-1}^n + kf_j^n, \end{aligned} \tag{3.5}$$

with $j \in \mathbb{Z}$, $n=0,1,2,\dots N-1$ and $v_j^0 = g_j = g(jh)$, $j \in \mathbb{Z}$ and $\lambda = \frac{k}{h^2}$. Define $\|v^n\|$ by

$$\|v^n\| = \sup_{j\in\mathbb{Z}} |v_j^n|, \quad n=0,1,2,\dots N$$

The values of v^n are essentially bounded in terms of the driving force f and the initial values g. We have the following theorem.

3.8 Theorem. *If the coefficients a,b,c, and f are bounded uniformly on the strip S_T and if $\inf_{S_T} a > 0$ then*

$$\exists\, \lambda_0 > 0\ \forall \lambda \in (0,\lambda_0)\ \exists\, h_\lambda > 0\ \forall\, h \in (0,h_\lambda) :$$

$$\|v^n\| \le \left(\sup_{\mathbb{R}} |g| + nk \sup_{S_T} |f|\right) e^{C_T nk}$$

where

$$C = \max\{0, \sup_{S_T} c\,\}$$

and v is a solution of our difference equation on the grid $\Sigma_{h,k}$ with $k=\lambda h^2$ and $0 \le nk \le T$.

Proof. We shall write $\sup a$ for $\sup_{S_T} a$ and $\inf a$ instead of $\inf_{S_T} a$. We shall write $\|f\|_\infty$ for $\sup_{S_T} |f|$ and $\|g\|_\infty$ instead of $\sup_{\mathbb{R}} |g|$.

By the triangle inequality

$$\begin{aligned} |v_j^{n+1}| \le \left|a_j^n\lambda + \frac{1}{2}b_j^n\lambda h\right| \|v^n\| + \left|1 - 2a_j^n\lambda + c_j^n\lambda h^2\right| \|v^n\| \\ + \left|a_j^n\lambda - \frac{1}{2}b_j^n\lambda h\right| \|v^n\| + k|f_j^n| \end{aligned}$$

Please note that we would experience a substantial cancellation of terms if the three leading coefficients were known to be positive. To this end we define λ_0 by

$$1 - 2\lambda_0 \sup a = 0 \Leftrightarrow \lambda_0 = \frac{1}{2\sup a},$$

and choose $\lambda < \lambda_0$. Then in particular

$$1 - 2\lambda a_j^n \geq 1 - 2\lambda \sup a > 1 - 2\lambda_0 \sup a = 0$$

for all values of j and n. Define $h_1(\lambda)$ by

$$\underbrace{1 - 2\lambda \sup a}_{\text{positive}} - \|c\|_\infty \lambda h_1(\lambda)^2 = 0.$$

Obviously $h_1(\lambda) > 0$. If we pick $h < h_1(\lambda)$ then

$$1 - 2\lambda a_j^n + c_j^n \lambda h^2 \geq 1 - 2\lambda \sup a - \|c\|_\infty \lambda h^2 > 0$$

Define h_2 by $\inf a - \frac{1}{2}\|b\|_\infty h_2 = 0$. Obviously $h_2 > 0$. If we pick $h < h_2$ then

$$\lambda a_j^n \pm \frac{1}{2}\lambda b_j^n h \geq \lambda \inf a - \frac{1}{2}\lambda \|b\|_\infty h > 0$$

for all values of j and n. Finally define $h_\lambda = \min\{h_1(\lambda), h_2\}$ and choose $h < h_\lambda$. Then

$$\begin{aligned}|v_j^{n+1}| &\leq \left(a_j^n \lambda + \frac{1}{2} b_j^n \lambda h + 1 - 2a_j^n \lambda + c_j^n \lambda h^2 + a_j^n \lambda - \frac{1}{2} b_j^n \lambda h\right) \|v^n\| + k\|f\|_\infty \\ &= (1 + c_j^n k)\|v^n\| + k\|f\|_\infty \leq (1 + Ck)\|v^n\| + k\|f\|_\infty,\end{aligned}$$

with $C \geq \sup_{S_T} c$. Therefore

$$\|v^{n+1}\| \leq (1 + Ck)\|v^n\| + k\|f\|_\infty.$$

If $C \neq 0$ then this may be iterated to yield

$$\|v^n\| \leq (1 + Ck)^n \|v^0\| + \frac{1}{C}\left((1 + Ck)^n - 1\right)\|f\|_\infty$$

Please note that if this estimate is correct for some value of n then

$$\begin{aligned}\|v^{n+1}\| &\leq (1 + Ck)\left[(1 + Ck)^n \|v^0\| + \frac{1}{C}\left((1 + Ck)^n - 1\right)\|f\|_\infty\right] + k\|f\|_\infty \\ &= (1 + Ck)^{n+1}\|v^0\| + \frac{1}{C}\big((1 + Ck)^{n+1} - (1 + Ck)\big)\|f\|_\infty + k\|f\|_\infty \\ &= (1 + Ck)^{n+1}\|v^0\| + \frac{1}{C}\big((1 + Ck)^{n+1} - 1\big)\|f\|_\infty\end{aligned}$$

and the estimate is valid for the next value of n. It is valid for $n = 0$. We may estimate $(1 + Ck)^n \leq e^{Cnk}$, provided that $Ck \geq -1$, so that

$$\begin{aligned}\|v^n\| &\leq e^{Cnk}\|v^0\| + \frac{1}{C}\left(e^{Cnk} - 1\right)\|f\|_\infty \\ &= e^{Cnk}\|v^0\| + nk\frac{e^{Cnk} - 1}{Cnk}\|f\|_\infty = e^{Cnk}\|v^0\| + nke^{\xi}\|f\|_\infty\end{aligned}$$

where ξ between 0 and Cnk is supplied by the mean value theorem. Now

$$\begin{aligned}C < 0 &\Rightarrow e^{\xi} \leq 1 \\ C > 0 &\Rightarrow e^{\xi} \leq e^{Cnk}.\end{aligned}$$

If $C = 0$ the estimate $\|v^n\| \leq \|v^0\| + nk\|f\|_\infty$ is immediate. Choosing

$$C = \max\{0, \sup_{S_T} c\}$$

forces $C \geq 0$ and provides the *general* estimate of

$$\|v^n\| \leq (\|g\|_\infty + nk\|f\|_\infty)\, e^{Cnk}.$$

3.9 Remark. Please note that this maximum principle immediately forces a uniqueness theorem for the difference equation. We may therefore speak of *the* solution of the difference equation on Σ_h with the existence being obvious from the equivalent iteration (3.5).

We now proceed essentially as before. We pick $\lambda < \lambda_0$ and $N \in \{1, 2, 3, \dots\}$ such that $k = T/N$ implies that $h = \sqrt{\frac{k}{\lambda}} < h_\lambda$ with λ_0 and h_λ being provided by theorem 3.8. Set $h_\nu = 2^{-\nu}h$ and $k_\nu = \lambda h_\nu^2$ and define the grids

$$\Sigma_\nu = \{(jh_\nu, nk_\nu) \mid j \in \mathbb{Z},\ 0 \leq nk_\nu \leq T\}$$

and $\Sigma = \cup_{\nu=0}^{\infty}\Sigma_\nu$. Let $v^\nu : \Sigma_\nu \to \mathbb{R}$ be the solution of equation (3.4) on Σ_ν. The task is to bound a number of the finite differences of v^ν uniformly and independently of ν. We begin by studying $\delta_x v^\nu$.

Let $v = v^\nu$, $h = h_\nu$ and $k = k_\nu$ for some large value of $\nu \geq \mu$, the value of which is to be determined later. Then

$$\begin{aligned}v_j^{n+1} = &\left(a\lambda + \frac{1}{2}b\lambda h\right) v_{j+1}^n + (1 - 2\lambda a + kc)\, v_j^n \\ &+ \left(a\lambda - \frac{1}{2}b\lambda h\right) v_{j-1}^n + kf\end{aligned}$$

with all evaluations of a, b, c, and f done at the point (jh, nk). Then

$$
\begin{aligned}
[\delta_x v]_j^{n+1} = & \left[E_x\left(a\lambda + \frac{1}{2}b\lambda h\right)\right]_j^n [\delta_x v]_{j+1}^n + \left[E_x\left(1 - 2\lambda a + kc\right)\right]_j^n [\delta_x v]_j^n \\
& + \left[E_x\left(a\lambda - \frac{1}{2}b\lambda h\right)\right]_j^n [\delta_x v]_{j-1}^n \\
& + \left[\delta_x\left(a\lambda + \frac{1}{2}b\lambda h\right)\right]_j^n v_{j+1}^n + \left[\delta_x\left(1 - 2\lambda a + kc\right)\right]_j^n v_j^n \\
& + \left[\delta_x\left(a\lambda - \frac{1}{2}b\lambda h\right)\right]_j^n v_{j-1}^n + k\delta_x f
\end{aligned}
$$

We focus on the three terms involving v_{j+1}^n, v_j^n and v_{j-1}^n. Their sum is

$$
\begin{aligned}
& \left[\delta_x\left(a\lambda + \frac{1}{2}b\lambda h\right)\right]_j^n v_{j+1}^n - \left[\delta_x\left(\lambda a\right)\right]_j^n v_j^n - \underbrace{\left[\delta_x \frac{1}{2}b\lambda h\right]_j^n v_j^n}_{\text{aux. term subtracted}} \\
& \quad + \left[\delta_x\left(a\lambda - \frac{1}{2}b\lambda h\right)\right]_j^n v_{j-1}^n - \left[\delta_x\left(\lambda a\right)\right]_j^n v_j^n + \underbrace{\left[\delta_x \frac{1}{2}b\lambda h\right]_j^n v_j^n}_{\text{and added again}} \\
& \qquad + k[\delta_x c]_j^n v_j^n.
\end{aligned}
$$

Writing $v_{j+1}^n - v_j^n = h[\delta_x v]_j^n$ and $v_{j-1}^n - v_j^n = -h[\delta_x v]_{j-1}^n$ gives

$$
\begin{aligned}
& h\left[\delta_x\left(a\lambda + \frac{1}{2}b\lambda h\right)\right]_j^n [\delta_x v]_j^n \\
& \quad - h\left[\delta_x\left(a\lambda - \frac{1}{2}b\lambda h\right)\right]_j^n [\delta_x v]_{j-1}^n \\
& \qquad + k[\delta_x c]_j^n v_j^n.
\end{aligned}
$$

We now return to the complete expression for $[\delta_x v]_j^{n+1}$ and obtain

$$
\begin{aligned}
[\delta_x v]_j^{n+1} = & \left[E_x\left(a\lambda + \frac{1}{2}b\lambda h\right)\right]_j^n [\delta_x v]_{j+1}^n \\
& + \left[E_x\left(1 - 2\lambda a + kc\right) + h\delta_x\left(a\lambda + \frac{1}{2}b\lambda h\right)\right]_j^n [\delta_x v]_j^n \\
& + \left[E_x\left(a\lambda - \frac{1}{2}b\lambda h\right) - h\delta_x\left(a\lambda - \frac{1}{2}b\lambda h\right)\right]_j^n [\delta_x v]_{j-1}^n \\
& + k[\delta_x c]_j^n v_j^n + k\delta_x f.
\end{aligned}
\tag{3.6}
$$

We now need to choose small values of h. Observe how the coefficient of $[\delta_x v]_j^n$ can be estimated from below

$$\begin{aligned}
&E_x(1-2\lambda a+kc)+h\left(\lambda\delta_x a+\frac{1}{2}\delta_x b\lambda h\right)\\
&\qquad\geq 1-2\lambda\|a\|_\infty-k\|c\|_\infty-h\lambda\|a_x\|_\infty-\frac{1}{2}\|b_x\|_\infty\lambda h^2\\
&\qquad\quad=1-2\lambda\|a\|_\infty-\underbrace{\left(\left\{\frac{1}{2}\|b_x\|_\infty+\|c\|_\infty\right\}\lambda h^2+\lambda h\|a_x\|_\infty\right)}_{\text{monotonic function of h}}.
\end{aligned}$$

We have already picked $\lambda<\frac{1}{2\|a\|_\infty}$ to achieve a bound for v_j^n, so that $1-2\lambda\|a\|_\infty$ is positive and we may now choose $0<h<h_0$ where h_0 satisfies the equation

$$1-2\lambda\|a\|_\infty-\left(\left\{\frac{1}{2}\|b_x\|+\|c\|_\infty\right\}\lambda h_0^2+\lambda h_0\|a_x\|_\infty\right)=0.$$

The third term of equation (3.6) is estimated in a similar fashion. Observe how

$$\begin{aligned}
&E_x\left(a\lambda-\frac{1}{2}b\lambda h\right)-h\delta_x\left(a\lambda-\frac{1}{2}b\lambda h\right)\\
&\qquad\geq\lambda\inf a-\frac{1}{2}\|b\|_\infty\lambda h-\|a_x\|_\infty\lambda h-\frac{1}{2}\|b_x\|_\infty\lambda h^2\\
&\qquad\quad=\lambda\inf a-\lambda\left(\frac{1}{2}\|b_x\|_\infty h^2+\left(\|a_x\|_\infty+\frac{1}{2}\|b\|_\infty\right)h\right)
\end{aligned}$$

We may pick $0<h<h_1$ where h_1 satisfies the equation

$$\lambda\inf a-\lambda\left(\frac{1}{2}\|b_x\|_\infty h_1^2+\left(\|a_x\|_\infty+\frac{1}{2}\|b\|_\infty\right)h_1\right)=0$$

Choosing $h<\min\{h_0,h_1\}$ ensures that the three coefficients of

$$[\delta_x v]_{j+1}^n,\quad[\delta_x v]_j^n,\quad[\delta_x v]_{j-1}^n$$

in equation (3.6) are positive. Then equation (3.6) implies that

$$|[\delta_x v]_j^{n+1}|\leq[1+kE_xc+k\delta_x b]_j^n\,\|\delta_x v^n\|+k\left|[(\delta_x c)v+\delta_x f]_j^n\right|,$$

so that

$$\|\delta_x v^{n+1}\|\leq(1+k(C+\|b_x\|_\infty)\,\|\delta_x v^n\|+k\left(\|c_x\|_\infty\|v^n\|+\|f_x\|_\infty\right)$$

with $C=\max\{0,\sup_{C_T}c\}$. For $0\leq nk\leq T$ we have the estimate

$$\|v^n\|\leq\underbrace{\left(\|g\|_\infty+T\|f\|_\infty\right)e^{CT}}_{\text{aux. term }W_T}$$

Then the estimate

$$\|\delta_x v^{n+1}\| \leq (1 + k(C + \|b_x\|_\infty)) \|\delta_x v^n\| + k (\|c_x\|_\infty W_T + \|f_x\|_\infty)$$

can be iterated to yield

$$\|\delta_x v^n\| \leq e^{(C+\|b_x\|_\infty)T} (\|g'\|_\infty + T (\|c_x\|_\infty W_T + \|f_x\|_\infty)),$$

for $0 \leq nk \leq T$.

We have made a few extra demands on the size of h, but with $h = h_\nu = 2^{-\nu} h_0$, these demands will be met for all $\nu \geq \mu$, provided that μ chosen is large enough. We have successfully bounded v^ν and $\delta_x v^\nu$ uniformly and independently of $\nu \geq \mu$. We still need to bound

$$\delta_t v, \delta_{xx} v, \delta_{xt} v, \delta_{tt} v, \delta_{xxx} v, \delta_{xtx} v$$

in a similar fashion. In view of theorem 3.4 the corresponding smooth problem is amazingly simple. Assume that all the functions a, b, c, f, and g are smooth with bounded derivatives and that u is a smooth solution of our differential equation

$$u_t = a u_{xx} + b u_x + c u + f, \quad (x,t) \in S_T$$

with $u(x,0) = g(x)$ for all $x \in \mathbb{R}$. Then $v = u_x$ is a solution of the equation

$$\begin{aligned} v_t = (u_x)_t = (u_t)_x &= (a u_{xx} + b u_x + c u + f)_x \\ &= a u_{xxx} + (a_x + b) u_{xx} + (b_x + c) u_x + c_x u + f_x \\ &= a v_{xx} + (a_x + b) v_x + (b_x + c) v + c_x u + f_x, \end{aligned}$$

where you should treat $\tilde{f} = c_x u + f_x$ as an inhomogeneous term with respect to v despite the obvious connection between u and v ! In short the derivative u_x satisfies an equation which is of the same type as the original one. The estimate

$$|u_x(x,t)| = |v(x,t)| \leq \left(\|g'\|_\infty + T\|\tilde{f}\|_\infty\right) e^{BT}, \quad (x,t) \in S_T$$

with $B = \max\{0, \sup(b_x + c)\}$, is inherited from the original equation. Inserting our estimate for u

$$|u(x,t)| \leq (\|g\|_\infty + T\|f\|_\infty) e^{CT}, \quad (x,t) \in S_T$$

will provide an a priori estimate for u_x, in terms of g, g', f, f_x, c, c_x, and b_x.

The derivative u_t can be estimated in a similar fashion. Observe how $w = u_t$ satisfies the equation

$$\begin{aligned} w_t = (u_t)_t &= (a u_{xx} + b u_x + c u + f)_t \\ &= a u_{xxt} + b u_{xt} + c u_t + f_t + a_t u_{xx} + b_t u_x + c_t u \end{aligned}$$

subject to the initial condition

$$w(x,0) = u_t(x,0) = a(x,0)g''(x) + b(x,0)g'(x) + c(x,0)g(x) + f(x,0)$$

Obviously we may bound w in terms of its initial values and the inhomogeneous term of $f_t + a_t u_{xx} + b_t u_x + c_t u$. We have already provided a priori bounds for u and u_x. u_{xx} is treated in a similar fashion. We merely need to determine the differential equation satisfied by u_{xx} and identify the inhomogeneous term as well as the creation term, i.e. the coefficient of (u_{xx}).

Our difference equation (3.4) suffers from the fact that the divided difference $\delta_x v$ satisfies an equation which is not of the same type as (3.4). Using the product rule of $\delta_x(fg) = (E_x f)\delta_x g + (\delta_x f)g$ we discover that

$$\begin{aligned}\delta_t(\delta_x v) = \delta_x(\delta_t v) &= \delta_x \left(a\delta_{x\overline{x}} + \frac{1}{2} b(\delta_x v + \delta_{\overline{x}} v) + cv + f \right) \\ &= (Ea)\delta_{x\overline{x}}(\delta_x v) + \frac{1}{2}(Eb)\delta_x(\delta_x v) + \frac{1}{2}(Eb)\delta_{\overline{x}}(\delta_x v) + (Ec)(\delta_x v) + \delta_x f \\ &\qquad + (\delta_x a)\delta_{\overline{x}}(\delta_x v) + \frac{1}{2}(\delta_x b)(\delta_x v) + \frac{1}{2}(\delta_x b)(\delta_{\overline{x}} v) + (\delta_x c)v\end{aligned}$$

We set $w = \delta_x v$ and group the coefficients of $\delta_{x\overline{x}} w, \delta_x w, \delta_{\overline{x}} w$ and w together, so that

$$\begin{aligned}\delta_t w = (Ea)\delta_{x\overline{x}} w &+ \frac{1}{2}(Eb)\delta_x w + \left(\frac{1}{2}(Eb) + (\delta_x a) \right) \delta_{\overline{x}} w \\ &+ \left(\frac{1}{2}(\delta_x b) + (Ec) \right) w + \frac{1}{2}(\delta_x b) E^{-1} w + (\delta_x f + (\delta_x c) v)\end{aligned}$$

Please note that the coefficients of $\delta_x w$ and $\delta_{\overline{x}} w$ need not be equal and in general they will be different and that an additional term of $(\delta_x b)E^{-1}w$ has surfaced. I have discovered at least two ways to avoid this problem. We might abandon the standard scheme and substitute it with

$$E_x \delta_t v = a\delta_{xx} v + b\delta_x v + cv + f,$$

with the curious shift E_x being needed in order to ensure stability. While this approach is beautiful it is of limited practical importance since the term $\delta_x v$ is only first order accurate in h. We shall not pursue this idea any further.

The second approach is to treat the standard scheme as a special case of the following equation

$$\delta_t v = a\delta_{x\overline{x}} v + b_1 \delta_x v + b_2 \delta_{\overline{x}} v + c_1 v + c_2 E_x^{-1} v + f \tag{3.7}$$

If v satisfies (3.7) then $w = \delta_x v$ satisfies the equation

$$\begin{aligned}\delta_t w = (E_x a)\delta_{x\overline{x}} w &+ (E_x b_1)\delta_x w + [(E_x b_2) + \delta_x a]\, \delta_{\overline{x}} w + [(E_x c_1) + \delta_x b_1]\, w \\ &+ [(E_x c_2) + \delta_x b_2]\, E_x^{-1} w + \left[\delta_x f + (\delta_x c_1) v + (\delta_x c_2) E_x^{-1} v \right].\end{aligned} \tag{3.8}$$

This equation is of the same type as equation (3.7). The initial condition for w is $w^0 = \delta_x g$. These observations are useless unless we can prove a maximum principle for (3.7). The equation is equivalent to

$$v_j^{n+1} = (a\lambda + b_1\lambda h)v_{j+1}^n + (1 - 2a\lambda - b_1\lambda h + b_2\lambda h + c_1\lambda h^2)v_j^n \\ + (a\lambda - b_2\lambda h + c_2\lambda h^2)v_{j-1}^n + kf_j^n$$

We can still experience the usual cancellations of terms and arrive at

$$|v_j^{n+1}| \leq (1 + (c_{1j}^n + c_{2j}^n)k)\|v^n\| + k|f_j^n|$$

provided that we choose $\lambda < \lambda_0$ and $h < h_\lambda$ with

$$1 - 2\sup a\lambda_0 = 0$$

and $h_\lambda = \min\{h_1, h_2(\lambda), h_3\}$ where

$$\inf a - \|b_1\|_\infty h_1 = 0$$
$$1 - 2\lambda \sup a - \lambda\|b_1 - b_2\|_\infty h_2(\lambda) - \lambda\|c_1\|_\infty h_2^2(\lambda) = 0$$
$$\inf a - \|b_2\|_\infty h_3 - \|c_2\|_\infty h_3^2 = 0$$

so that the three coefficients of $u_{j+1}^n, u_j^n, u_{j-1}^n$ are sure to be positive. As in the proof of theorem 3.8 we choose $C = \max\{0, \sup(c_1 + c_2)\}$ and proceed in the same fashion with

$$\|v^{n+1}\| \leq (1 + Ck)\|v^n\| + k\|f\|_\infty$$

and by induction on n

$$\|v^n\| \leq \left(\|v^0\| + nk\|f\|_\infty\right) e^{Cnk}.$$

Notice that the inhomogeneous term of equation (3.8), that is the term

$$\left[\delta_x f + (\delta_x c_1)v + (\delta_x c_2)E_x^{-1}v\right]$$

involves v, but that only an estimate of the maximum size of the term is needed in order to estimate w. This is easily done by combining the triangle inequality with the estimate of v. The value of λ needed to force stability of v can be applied to w, since the choice of $\lambda < \lambda_0$ depends exclusively on the leading coefficient of the right hand side of the difference equation (3.7)

$$\lambda_0 = \frac{1}{2\sup a} = \frac{1}{2\sup(E_x a)}.$$

The maximum principle can be applied successfully as long as all the coefficients of (3.7) as well as the initial condition are bounded independent of the grid. Let Δ^p be the class of continuous functions for which all divided differences of order less than or equal to p are bounded independent of the grid. If g is p times differentiable with bounded derivatives, then $g \in \Delta^p$. If $g \in \Delta^{p+1}$ then the first order divided differences of g belong to Δ^p. If the coefficients of the difference equation of v as well as the initial condition for v belong to Δ^{p+1} then the coefficients and the initial condition for w belong to Δ^p. We are essentially losing one degree of differentiability.

We summarize this discussion and state the similar result for $q = \delta_t v$ in the following

3.10 Proposition. *Let $a, b_i, c_i, f \in \Delta^{p+2}$, and let $g \in \Delta^{p+2}(\mathbb{R})$. If the grid function $v = v_{h,k} : \Sigma_{h,k} \to \mathbb{R}$ satisfies the equation (3.7) with an initial condition of $v_j^0 = v(jh, 0) = g(jh)$ then $w = w_{h,k} = \delta_x v_{h,k}$ satisfies the equation*

$$\begin{aligned}\delta_t w = (E_x a)\delta_{x\overline{x}} w + (E_x b_1)\delta_x w + [(E_x b_2) + \delta_x a]\, \delta_{\overline{x}} w + [(E_x c_1) + \delta_x b_1]\, w \\ + [(E_x c_2) + \delta_x b_2]\, E_x^{-1} w + \left[\delta_x f + (\delta_x c_1) v + (\delta_x c_2) E_x^{-1} v\right],\end{aligned}$$

with the initial condition of $w_j^0 = \delta_x g(jh)$ and the coefficients of this equation along with the initial condition are members of Δ^{p+1}.

Similarly $q = q_{h,k} = \delta_t v_{h,k}$ satisfies the equation

$$\begin{aligned}\delta_t q = (E_t a)\delta_{x\overline{x}} q + (E_t b_1)\delta_x q + (E_t b_2)\delta_{\overline{x}} q + (E_t c_1) q + (E_t c_2) E_x^{-1} q \\ + \left[\delta_t f + (\delta_t a)\delta_{x\overline{x}} v + (\delta_t b_1)\delta_x v + (\delta_t b_2)\delta_{\overline{x}} v + (\delta_t c_1) v + (\delta_t c_2) E_x^{-1} v\right],\end{aligned}$$

with the initial condition of

$$q^0 = a\delta_{x\overline{x}} g + b_1 \delta_x g + b_2 \delta_{\overline{x}} g + c_1 g + c_2 E^{-1} g + f^0,$$

but the coefficients and the initial condition are only *members of Δ^p. Two degrees of differentiability are lost.*

Proof. The proof is elementary. It is merely a matter of keeping track of the various terms.

We were originally given the task of bounding the divided differences of v.

Suppose that a, b, c, f, and g are $2m$ times differentiable with bounded derivatives. Then we derive the equations satisfied by $\delta_x^j v, j = 1, 2, \ldots, 2m$ iteratively and estimate $\delta_x^{j+1} v$ using the maximum principle and the previous estimates for $\delta_x^{j'} v$, $j' = 0, 1, 2, \ldots, j$. This procedure is repeated for $w = \delta_t v$, but it must terminate at $j = 2m - 2$, because we have already used up two degrees of differentiability. We may continue this process and obtain a priori estimates for $\delta_t^i \delta_x^j v, i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, 2m - 2i$. The λ_0 given by the maximum principle is independent of i and j and each equation imposes a bound on the step size h, but these will eventually be met by $h_\nu = 2^{-\nu} h_0$.

If we desire a function in $C^p(S_T)$ we must bound all divided differences of order less than or equal to $p + 1$. In particular we must bound $\delta_t^{p+1} v$ independently of the grid. It therefore suffices to have a, b, c, f, and g $2p + 2$ times differentiable with bounded derivatives. Using the machinery of chapter 2 we may then generate a function in $C^p(S_T)$ as our candidate for a solution to our differential equation.

Obviously if a, b, c, f, and g are infinitely differentiable with bounded derivatives then our candidate will be a member of $C^\infty(S_T)$.

In our case we do not quite want a C^2 function but are satisfied with bounding

$$\delta_t^i \delta_x^j v, \quad i = 0, 1, 2 \quad j = 0, \ldots, 4 - 2i$$

It is therefore sufficient to have a, b, c, f, and g four times differentiable with bounded derivatives.

3.11 Remark. Technically $\delta_t v^\nu$ is not directly defined on the final line of $t = T$. We use the differential equation to define $\delta_t v^\nu$ there.

We may now proceed as in the case of the heat equation and define functions $u, u', u'', \dot{u} : \Sigma \to \mathbb{R}$ through the usual relations of

$$u(x,t) = \lim_{\nu\to\infty} v^\nu(x,t) \qquad u'(x,t) = \lim_{\nu\to\infty} [\delta_x v^\nu](x,t)$$
$$u''(x,t) = \lim_{\nu\to\infty} [\delta_{xx} v^\nu](x,t) \qquad \dot{u}(x,t) = \lim_{\nu\to\infty} [\delta_t v^\nu](x,t)$$

These functions are all densely defined, bounded and Lipschitz continuous functions and as such admit unique extensions to the entire domain of S_T. We claim that u satisfies the differential equation. Let $(x,t) \in \Sigma$. Then

$$\delta_{\overline{x}} v^\nu(x,t) = \delta_x v^\nu(x - h_\nu, t) = \delta_x v^\nu(x,t) - h_\nu \delta_{xx} v^\nu(x - h_\nu, t) \to u'(x,t)$$

because $\delta_{xx} v^\nu(x - h_\nu, t)$ is bounded independently of ν. Thus

$$\mu_x v^\nu = \frac{1}{2}\left(\delta_x v^\nu + \delta_{\overline{x}} v^\nu\right) \to u'$$

Similarly

$$\delta_{x\overline{x}} v^\nu(x,t) = \delta_{xx} v^\nu(x - h_\nu, t) = \delta_{xx} v^\nu(x,t) - \underbrace{\delta_{xxx} v^\nu(x - h_\nu, t)}_{\text{bounded}} h_\nu \to u''(x,t)$$

so that

$$a\delta_{x\overline{x}} v^\nu + b\mu_x v^\nu + cv^\nu + f \to au'' + bu' + cu + f.$$

But $\delta_t v^\nu \to \dot{u}$ so that

$$\dot{u} = au'' + bu' + cu + f$$

for all $(x,t) \in \Sigma$. Recalling that u is differentiable with $u_t = \dot{u}, u_x = u'$ and u_x is differentiable with respect to x with $u_{xx} = u''$ we discover that the differential equation is in fact satisfied for all (x,t) in the dense set Σ. By continuity it is satisfied everywhere.

The initial condition is automatically satisfied because by construction u agrees with the continuous function g on a dense subset of the initial line.

This completes the proof of theorem 3.3.

We now turn to the proof of theorem 3.4. We shall need

3.12 Lemma. *If $u : S_T \to \mathbb{R}$ is a solution of our differential equation, and if u_x, u_t, and u_{xx} are uniformly continuous then the explicit method converges uniformly to u on S_T and u inherits the maximum principle*

$$|u(x,t)| \le (\|g\|_\infty + T\|f\|_\infty)\, e^{CT}, \quad (x,t) \in S_T$$

with $C = \max\{0, \sup_{S_T} c\}$

Proof. Let v^ν be the solution of the difference equation on the grid Σ_ν. Then $u - v^\nu$ vanishes in all grid points on the initial line and satisfies the difference equation

$$\begin{aligned}\delta_t(u - v^\nu) &= u_t + (\delta_t u - u_t) - \delta_t v^\nu \\ &= a u_{xx} + b u_x + cu + f + (\delta_t u - u_t) - (a\delta_{x\overline{x}} v^\nu + b\mu_x v^\nu + cv^\nu + f) \\ &= a\delta_{x\overline{x}}(u - v^\nu) + b\mu_x(u - v^\nu) + c(u - v^\nu) \\ &\quad + \underbrace{a(u_{xx} - \delta_{x\overline{x}} u) + b(u_x - \mu_x u) + (\delta_t u - u_t)}_{\text{inhomogeneous term}}\end{aligned}$$

Given $\epsilon > 0$ the uniform continuity of u_t, u_x, u_{xx} along with the boundedness of a and b ensures the existence of μ such that for all $\nu \ge \mu$

$$|a(u_{xx} - \delta_{x\overline{x}} u) + b(u_x - \mu_x u) + (\delta_t u - u_t)| < \epsilon$$

Then the discrete maximum principle of theorem 3.8 yields the estimate

$$|u - v^\nu| \le \epsilon e^{CT},$$

for all $(x,t) \in \Sigma_h$ with $0 \le nk \le T$. Trivially

$$\begin{aligned}|u(x,t)| &\le |v^\nu(x,t)| + |u(x,t) - v^\nu(x,t)| \\ &\le (\|g\|_\infty + T\|f\|_\infty)\, e^{CT} + \epsilon e^{CT}\end{aligned}$$

for all $(x,t) \in \Sigma_\nu$, with $\nu \ge \mu$. Letting ϵ tend to zero yields the estimate

$$|u(x,t)| \le (\|g\|_\infty + T\|f\|_\infty)\, e^{CT}$$

for all $(x,t) \in \Sigma \cap S_T$, which is readily extended to the entire strip by the continuity of u.

3.13 Remark. This proof is identical to the one found in Fritz John's book, [1]. John however assumed that u should also be uniformly continuous, but did not make use of it. This is hardly important since we are working mainly with functions with bounded derivatives, so that they are not only uniformly continuous but even uniformly Lipschitz continuous.

Following Fritz John we proceed to eliminate the assumption of uniform continuity

3.14 Lemma. *If u solves the initial value problem and if u, u_x, u_t and u_{xx} are continuous and if u, u_x are bounded uniformly on S_T then*

$$|u(x,t)| \leq (\|g\|_\infty + T\|f\|_\infty)\, e^{CT}$$

for all $(x,t) \in S_T$.

Proof. We use a standard trick and reduce this case to the previous one by means of a smooth cut-off function $\phi : \mathbb{R} \to [0,1]$ with compact support within $[-2,2]$ and $\phi \equiv 1$ on $[-1,1]$. Define ψ^n by

$$\psi^n(x,t) = \phi\left(\frac{x}{n}\right) u(x,t)$$

Then ψ_n and all its derivatives are compactly supported and are therefore uniformly continuous. Elementary calculations yield

$$\begin{aligned}
\psi^n_x &= u_x\phi\left(\frac{x}{n}\right) + u\phi'\left(\frac{x}{n}\right)\frac{1}{n} \\
\psi^n_{xx} &= u_{xx}\phi\left(\frac{x}{n}\right) + 2u_x\phi'\left(\frac{x}{n}\right)\frac{1}{n} + u\phi''\left(\frac{x}{n}\right)\frac{1}{n^2} \\
\psi^n_t &= u_t\phi\left(\frac{x}{n}\right)
\end{aligned}$$

so that ψ satisfies the equation

$$\begin{aligned}
&\psi^n_t - (a\psi^n_{xx} + b\psi^n_x + c\psi^n) \\
&\qquad = (u_t - (au_{xx} + bu_x + cu))\,\phi\left(\frac{x}{n}\right) + R_n(x,t) \\
&\qquad\qquad = f\phi\left(\frac{x}{n}\right) + R_n(x,t)
\end{aligned}$$

where the remainder R_n is

$$R_n = -bu\phi'\left(\frac{x}{n}\right)\frac{1}{n} - a\left(2u_x\phi'\left(\frac{x}{n}\right)\frac{1}{n} + u\phi''\left(\frac{x}{n}\right)\frac{1}{n^2}\right).$$

By assumption u and u_x are bounded so there exists a constant B independent of n, x, t such that

$$|R_n(x,t)| \leq \frac{B}{n}.$$

Applying the previous lemma to ψ^n yields the estimate

$$|\psi^n(x,t)| \leq \left(\sup_{\mathbb{R}} |u(x,0)| + \left(\sup_{S_T} |f| + \frac{B}{n}\right)\right) e^{CT}.$$

Since $\psi^n(x,t) \to u(x,t)$ for all (x,t) we are left with the conclusion that

$$|u(x,t)| \leq \left(\sup_{\mathbb{R}} |u(x,0)| + \sup_{S_T} |f|\right) e^{CT}.$$

3.15 Remark. This lemma is the promised theorem 3.4.

This lemma immediately settles the question of uniqueness of solutions to our initial value problem. If u and v are two solutions with u, u_x, u_t continuous and u, u_x bounded and similarly for v then the difference is subject to the estimate

$$|(u-v)(x,t)| \leq \left(\sup_{\mathbb{R}} |(u-v)(x,0)| \right) e^{CT} = 0$$

since the differential equation for $u - v$ does not have an inhomogeneous term and $u - v \equiv 0$ on the initial line. This concludes the proof of theorem 3.5.

Please note that the construction of our solution is independent of λ and h_0. Before establishing the uniqueness theorem there was no way of telling if our solution depended on these parameters in some subtle way. But the solutions generated by the finite difference method all satisfy the requirements of the uniqueness theorem.

The following global theorem is immediate. Let $\Omega = \mathbb{R} \times (0, \infty)$.

3.16 Theorem. *Let $a, b, c, f : \overline{\Omega} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be given. If g is bounded and four times differentiable with bounded derivatives and if $a, b, c,$ and f are bounded and four times differentiable with bounded derivatives on every strip S_T and if $\inf_{S_T} a > 0$ on every strip S_T then there exists a unique solution $u : \overline{\Omega} \to \mathbb{R}$ of equation* (3.1) *so that u, u_x, u_t and u_{xx} are bounded and Lipschitz continuous on every strip S_T. In addition u is subject to the following estimate*

$$|u(x,t)| \leq (\|g\|_\infty + t\|f\|_{t,\infty})\, e^{C_t t}$$

where

$$C_t = \max\{0, \sup_{S_t} c\} \quad \text{and} \quad \|f\|_{t,\infty} = \sup_{S_t} |f|.$$

Proof. The proof is obvious. Let $T > 0$ and let $u_T : S_T \to \mathbb{R}$ be the solution of the initial value problem (3.1). We define $u : \overline{\Omega} \to \mathbb{R}$ by

$$u(x,t) = u_T(x,t)$$

for all $(x,t) \in S_T$. Is u well-defined ? Let $0 < T_1 < T_2$ be given. Then by the uniqueness theorem $u_{T_1} \equiv u_{T_2}$ on S_{T_1} and u is indeed well-defined. Similarly if u and v are two solutions to the initial value problem then by the uniqueness theorem 3.5 $u - v \equiv 0$ on every strip S_T. As for the estimate we have

$$|u(x,t)| = |u_T(x,t)| \leq (\|g\|_\infty + T\|f\|_{T,\infty})\, e^{C_T T}, \quad (x,t) \in S_T$$

from theorem 3.4, where

$$C_T = \max\{0, \sup_{S_T} c\} \quad \text{and} \quad \|f\|_{T,\infty} = \sup_{S_T} |f|.$$

In particular we have for $T \geq 0$ that

$$|u(x,T)| \leq (\|g\|_\infty + T\|f\|_{T,\infty})\, e^{C_T T}, \quad x \in \mathbb{R}.$$

Replacing T with t completes the proof.

3.17 Remark. Please note that we may have $\inf_{\overline{\Omega}} a = 0$ as long as $\inf_{S_T} a > 0$ for every strip S_T. The estimate of theorem 3.16 emphasizes the fact that the solution is independent of future values of a, b, c, and f.

Lemma 3.12 stated that the explicit method actually converges to the solution under quite general circumstances. It is possible to be more precise. Let $u : S_T \to \mathbb{R}$ be a smooth solution of our initial value problem such that u and all its derivatives are bounded and let v_h be the solution of the difference equation (3.2) on the grid Σ_h. By Taylor expansion

$$\begin{aligned} \delta_t u &= u_t + \frac{1}{2}u_{tt}(x,\tau)k \\ \mu_x u &= u_x + \frac{1}{6}u_{xxx}(\xi,t)h^2 \\ \delta_{x\overline{x}} u &= u_{xx} + \frac{1}{12}u_{xxxx}(\nu,t)h^2 \end{aligned}$$

for appropriately chosen τ, ξ and ν depending on (x,t). Then $u - v_h$ satisfy the difference equation

$$\begin{aligned} \delta_t(u - v_h) &= u_t + \frac{1}{2}u_{tt}(x,\tau)k - \delta_t v_h \\ &= au_{xx} + bu_x + cu + f - (a\delta_{x\overline{x}}v_h + b\mu_x v_h + cv_h + f) + \frac{1}{2}u_{tt}(x,\tau)k \\ &= a\delta_{x\overline{x}}(u - v_h) + b\mu_x(u - v_h) + c(u - v_h) \\ &\qquad + a(u_{xx} - \delta_{x\overline{x}}u) + b(u_x - \mu_x u) + \frac{1}{2}u_{tt}(x,\tau)k. \end{aligned}$$

The final term is quite small

$$\begin{aligned} &a(u_{xx} - \delta_{x\overline{x}}u) + b(u_x - \mu_x u) + \frac{1}{2}u_{tt}(x,\tau)k \\ &\qquad = -\frac{1}{12}au_{xxxx}(\nu,t)h^2 - \frac{1}{6}bu_{xxx}(\xi,t)h^2 + \frac{1}{2}u_{tt}(x,\tau)\lambda h^2 \\ &\qquad = O(h^2), \end{aligned}$$

and since $u - v_h \equiv 0$ on the initial line we may apply the discrete maximum principle of theorem 3.8 and gain the estimate

$$|u(x,t) - v_h(x,t)| \leq C(u,T)h^2 \tag{3.9}$$

for all $(x,t) \in \Sigma_h \cap S_T$. The constant $C(u,T)$ obviously depends on the exact solution which we are trying to calculate. This inequality is apparently useless in practical error estimation, but it establishes the rate of convergence. Estimate (3.9) is however crucial in establishing a Taylor-like error expansion as we shall demonstrate in chapter 4. The estimate is valid as long as h is chosen so small that theorem 3.8 applies.

For easy reference we state the result as a lemma.

3.18 Lemma. *Let $u : S_T \to \mathbb{R}$ be a smooth solution of the initial value problem* (3.1) *such that u and all its derivatives are bounded on S_T. Then there exists a constant $C(u,T)$ depending exclusively on u and T such that*

$$|u(x,t) - v_h(x,t)| \leq C(u,T)h^2$$

for all $(x,t) \in S_T \cap \Sigma_h$, where $v_h : \Sigma_h \to \mathbb{R}$ is the solution of the corresponding explicit finite difference equation (3.2) *and the grid Σ_h is determined by $\lambda < \lambda_0$, $h < h_0$ and $k = \lambda h^2$, with λ_0 and h_0 being supplied by theorem 3.8.*

3.19 Remark. This is the prototypical theorem of numerical analysis applied to differential equations. If a smooth solution exists then your scheme converges to it with a speed that depends on the amount of work you put into it.

3.20 Remark. $u : S_T \to \mathbb{R}$ does not really need to be smooth, i.e. infinitely differentiable, we just need enough differentiability to be able to perform the Taylor expansions. In the special case of the heat equation the maximum principle of theorem 3.8 remains valid if $\lambda \leq \lambda_0 = \frac{1}{2}$ and no restrictions are placed on h and lemma 3.18 can be adjusted accordingly.

We conclude this chapter with a series of easy theorems that show that the solution u depends continuously on the initial condition g, the driving force f and the coefficients a, b and c of the differential equation. They are not relevant for the rest of the thesis but serve as an illustration of the power of maximum principles.

For simplicity we assume that a, b, c, and $f : S_T \to \mathbb{R}$ are bounded on S_T, infinitely differentiable with derivatives that are bounded on S_T and that $g : \mathbb{R} \to \mathbb{R}$ is bounded and infinitely differentiable with bounded derivatives.

Consider a family of functions $w(\xi) : A \to \mathbb{R}$, on some set A, with $\xi \in \mathbb{R}$. We shall only need $A = \mathbb{R}$ or $A = S_T$.

3.21 Definition. *We say that $w(\xi) \to w(\xi_0)$ uniformly on A as $\xi \to \xi_0$ iff*

$$\sup_{x \in A} |w(x;\xi) - w(x;\xi_0)| \to 0, \quad \text{as} \quad \xi \to \xi_0.$$

The following lemma is an immediate consequence of the smooth maximum principle

3.22 Lemma. *If $g(\xi) \to g(\xi_0)$ uniformly on $\mathbb{R}$ and if $f(\xi) \to f(\xi_0)$ uniformly on S_T then the solutions $u(\xi)$ of the differential equations*

$$u_t(x,t;\xi) = a(x,t)u_{xx}(x,t;\xi) + b(x,t)u_x(x,t;\xi) + c(x,t)u(x,t;\xi) + f(x,t;\xi)$$

for $(x,t) \in S_T$ with $u(x,0;\xi) = g(x;\xi)$ converge uniformly to $u(\xi_0)$ on S_T.

Proof. We must estimate the size of the function $u(\xi) - u(\xi_0)$. Obviously

$$\begin{aligned}(u(\xi) - u(\xi_0))_t &= au_{xx}(\xi) + bu_x(\xi) + cu(\xi) + f(\xi) \\ &- au_{xx}(\xi_0) - bu_x(\xi_0) - cu(\xi_0) - f(\xi_0) \\ &= a(u(\xi) - u(\xi_0))_{xx} + b(u(\xi) - u(\xi_0))_x \\ &+ c(u(\xi) - u(\xi_0)) + f(\xi) - f(\xi_0)\end{aligned}$$

for all $t > 0$ and $u(\xi) - u(\xi_0) = g(\xi) - g(\xi_0)$ on the initial line of $t = 0$. By the smooth maximum principle we have

$$|u(x,t;\xi) - u(x,t;\xi_0)| \leq \left(\sup_{x\in\mathbb{R}}|g(x;\xi) - g(x;\xi_0)| + T \sup_{(x,t)\in S_T} |f(x,t;\xi) - f(x,t;\xi_0)|\right) e^{C_T T}$$

which tends to zero as ξ tends to ξ_0 independently of $(x,t) \in S_T$.

Before considering the more general case of a, b, and c depending on ξ we need the following lemma.

3.23 Lemma. *Let $w(\xi) : A \to R$ be a family of functions on some set A, with $\xi \in \mathbb{R}$. If $w(\xi)$ is bounded for every ξ and if $w(\xi)$ converges uniformly to $w(\xi_0)$ on A then the function*

$$\xi \to \sup_{x\in A} w(x;\xi)$$

is defined for every ξ and continuous at ξ_0.

Proof. The proof is trivial but is included anyway for the sake of completeness. Let $\epsilon > 0$ be given. Then there exists $\delta > 0$ such that

$$|w(x;\xi) - w(x;\xi_0)| < \epsilon$$

for all $x \in A$, provided that $|\xi - \xi_0| < \delta$. Pick ξ such that $|\xi - \xi_0| < \delta$. Then

$$\begin{aligned}w(x;\xi_0) = w(x;\xi) + (w(x;\xi_0) - w(x;\xi)) \\ < w(x;\xi) + \epsilon \leq \sup_{x\in A} w(x;\xi) + \epsilon\end{aligned}$$

Therefore

$$\sup_{x\in A} w(x;\xi_0) \leq \sup_{x\in A} w(x;\xi) + \epsilon$$

and interchanging ξ and ξ_0 completes the proof.

3.24 Corollary. *If $g(\xi) \to g(\xi_0)$ uniformly on $\mathbb{R}$, $f(\xi) \to f(\xi_0)$ uniformly on S_T and if $c(\xi) \to c(\xi_0)$ uniformly on S_T then it is possible to bound $u(\xi)$ on S_T independently of ξ for all ξ in some neighbourhood of ξ_0.*

Proof. By the smooth maximum principle the functions $u(\xi)$ are bounded in the following fashion

$$|u(x,t;\xi)| \leq (\|g(\xi)\|_\infty + T\|f(\xi)\|_{T,\infty})\, e^{C_T(\xi)}$$

for all $(x,t) \in S_T$. Our task is to bound the right hand side independently of ξ. Consider the function $\|g(\xi)\|_\infty : \mathbb{R} \to \mathbb{R}$ given by

$$\|g(\xi)\|_\infty = \sup_{x\in R} |g(x;\xi)|.$$

By the triangle inequality $|g(\xi)| \to |g(\xi_0)|$ uniformly on $\mathbb{R}$ and by the previous lemma our function $\xi \to \|g(\xi)\|_\infty$ is continuous at ξ_0. Similar reasoning establishes the fact that

$$\xi \to \|f(\xi)\|_{T,\infty} = \sup_{(x,t)\in S_t} |f(x,t;\xi)|$$
$$\xi \to C_T(\xi) = \max\{0, \sup_{(x,t)\in S_T} c(x,t;\xi)\}$$

are continuous at ξ_0. Now pick an $\epsilon > 0$ at random. By continuity there exists $\delta > 0$ such that

$$\|g(\xi)\|_\infty < \|g(\xi_0)\|_\infty + \epsilon$$
$$\|f(\xi)\|_{T,\infty} < \|f(\xi_0)\|_{T,\infty} + \epsilon$$
$$C_T(\xi) < C_T(\xi_0) + \epsilon$$

for all ξ with $|\xi - \xi_0| < \delta$. Therefore

$$|u(x,t;\xi)| \leq (\|g(\xi_0)\|_\infty + \epsilon + T(\|f(\xi_0)\|_{T,\infty} + \epsilon))\, e^{(C_T(\xi_0)+\epsilon)T}$$

for all $(x,t) \in S_T$ provided $|\xi - \xi_0| < \delta$.

We are ready for the final theorem.

3.25 Theorem. *If $D_x^j g(\xi) \to D_x^j(\xi_0)$ uniformly on $\mathbb{R}$ for $j = 0,1,2$ and if*

$$D_x^j a(\xi) \to D_x^j a(\xi_0) \qquad D_x^j b(\xi) \to D_x^j b(\xi_0),$$
$$D_x^j c(\xi) \to D_x^j c(\xi_0) \qquad D_x^j f(\xi) \to D_x^j f(\xi_0)$$

uniformly on S_T for each $j = 0,1,2$, then

$$u(\xi) \to u(\xi_0)$$

uniformly on S_T as $\xi \to \xi_0$.

Proof. We must compare $u(\xi)$ to $u(\xi_0)$ on S_T. Now by definition

$$(u(\xi) - u(\xi_0))_t = a(\xi)u_{xx}(\xi) + b(\xi)u_x(\xi) + c(\xi)u(\xi) + f(\xi)$$
$$- a(\xi_0)u_{xx}(\xi_0) - b(\xi_0)u_x(\xi_0) - c(\xi_0)u(\xi_0) - f(\xi_0).$$

We rewrite $a(\xi)u_{xx}(\xi) - a(\xi_0)u_{xx}(\xi_0)$ as

$$a(\xi)u_{xx}(\xi) - a(\xi_0)u_{xx}(\xi_0) = a(\xi_0)\left(u(\xi) - u(\xi_0)\right)_{xx} + \left(a(\xi) - a(\xi_0)\right)u_{xx}(\xi)$$

and similarly for the terms involving b and c to get

$$\begin{aligned}
&(u(\xi) - u(\xi_0))_t \\
&\quad = a(\xi_0)\left(u(\xi) - u(\xi_0)\right)_{xx} + b(\xi_0)\left(u(\xi) - u(\xi_0)\right)_x + c(\xi_0)\left(u(\xi) - u(\xi_0)\right) \\
&\qquad + \left(a(\xi) - a(\xi_0)\right)u_{xx}(\xi) + \left(b(\xi) - b(\xi_0)\right)u_x(\xi) + \left(c(\xi) - c(\xi_0)\right)u(\xi) \\
&\qquad\qquad + f(\xi) - f(\xi_0)
\end{aligned}$$

along with the initial condition of $u(\xi) - u(\xi_0) = g(\xi) - g(\xi_0)$. Now in order to apply the maximum principle successfully we must gain control of the complicated inhomogeneous term of

$$\begin{aligned}
&f(\xi) - f(\xi_0) + \left(a(\xi) - a(\xi_0)\right)u_{xx}(\xi) \\
&\qquad + \left(b(\xi) - b(\xi_0)\right)u_x(\xi) + \left(c(\xi) - c(\xi_0)\right)u(\xi)
\end{aligned}$$

Only the terms of $u(\xi), u_x(\xi), u_{xx}(\xi)$ present any problems. By the previous lemma it is possible to bound $u(\xi)$ independently of ξ for all ξ within some neighbourhood of ξ_0. Recall that if u solves our initial value problem of

$$u_t - au_{xx} - bu_x - cu = f$$

with $u(x, 0) = g(x)$, then

$$\begin{aligned}
(u_x)_t &= a(u_x)_{xx} + (a_x + b)(u_x)_x + (b_x + c)(u_x) + c_x u + f_x \\
(u_{xx})_t &= a(u_{xx})_{xx} + (2a_x + b)(u_{xx})_x + (a_{xx} + 2b_x + c)(u_{xx}) \\
&\quad + (b_{xx} + 2c_x)u_x + c_{xx}u + f_{xx}
\end{aligned}$$

with the initial conditions of $u_x(x, 0) = g'(x)$ and $u_{xx}(x, 0) = g''(x)$, respectively. Now focus on the first of these two equations. By applying the bound on u along with the previous lemmata we may bound the inhomogeneous term of $c_x u + f_x$ along with the creation term of $(b_x + c)$ independently of ξ for all ξ within some possibly smaller neighbourhood of ξ_0. Thus we obtain a bound for $u_x(\xi)$ on S_T which is independent of ξ and valid for all ξ within a neighbourhood of ξ_0. We may repeat this procedure for $u_{xx}(\xi)$. In summary: There exists $\delta_1 > 0$ and a constant M such that for all ξ with $|\xi - \xi_0| < \delta_1$ we have

$$\|D_x^j u\|_{T,\infty} < M, \quad j = 0, 1, 2$$

Now given $\epsilon > 0$ we have by assumption a $\delta_2 > 0$ such that

$$\begin{aligned}
\|f(\xi) - f(\xi_0)\| &< \frac{\epsilon}{4} \qquad \|a(\xi) - a(\xi_0)\| < \frac{\epsilon}{4M} \\
\|b(\xi) - b(\xi_0)\| &< \frac{\epsilon}{4M} \qquad \|c(\xi) - c(\xi_0)\| < \frac{\epsilon}{4M}
\end{aligned}$$

for all ξ with $|\xi - \xi_0| < \delta_2$. Choosing $\delta = \min\{\delta_1, \delta_2\}$ and ξ such that $|\xi - \xi_0| < \delta$ provides the estimate of

$$\begin{aligned} -\epsilon < f(\xi) - f(\xi_0) + [a(\xi) - a(\xi_0)]\, u_{xx}(\xi) \\ + [b(\xi) - b(\xi_0)]\, u_x(\xi) + [c(\xi) - c(\xi_0)]\, u(\xi) < \epsilon. \end{aligned}$$

The smooth maximum principle gives

$$|u(\xi) - u(\xi_0)| \leq (|g(\xi) - g(\xi_0)| + T\epsilon)\, e^{C_T(\xi_0)T}$$

and we are done.

3.26 Remark. We do not need that $a_x(\xi) \to a_x(\xi_0)$ uniformly on S_T, but that is not really important. The purpose of these final theorems was to illustrate yet another use of maximum principles. I was not looking for “sharp” results.

Chapter 4

Error Analysis

Let $\Omega = \mathbb{R} \times (0, \infty)$ and let $Pu = u_t - u_{xx}$ and consider once again the pure initial value problem of

$$\begin{aligned} Pu &= f, \quad (x,t) \in \Omega \\ u &= g, \quad t = 0 \end{aligned}$$

along with the corresponding discrete problem of

$$\begin{aligned} P_h v_h &= f, \quad (x,t) \in \Sigma_h \\ v_h^0 &= g \end{aligned}$$

where $P_h v_h = \delta_t v_h - \delta_{x\overline{x}} v_h$, with $k = \lambda h^2$ and $\lambda \leq \frac{1}{2}$. We assume that g is bounded and infinitely differentiable with bounded derivatives and that f is infinitely differentiable and that f along with all its derivatives are bounded on every strip

$$S_T = \{(x,t) \,:\, x \in \mathbb{R},\, 0 \leq t \leq T\}.$$

In this case theorem 3.16 establishes that our differential equation has a unique solution u which is bounded and infinitely differentiable with bounded derivatives on every strip S_T.

In this chapter we establish the existence of an error expansion of the desired form

$$u(x,t) - v_h(x,t) = \alpha(x,t)h^2 + \beta(x,t)h^4 + \gamma(x,t)h^6 + \ldots$$

The coefficients $\alpha, \beta, \gamma \ldots$ are independent of h and are given as solutions to differential equations similar to the original one, but with increasingly complicated inhomogeneous terms.

We begin with the following lemma which is merely yet another application of Taylor's theorem.

4.1 Lemma. *(Basic expansion.) For all (x,t) there exists $\xi \in (x-h, x+h)$ and $\tau \in (t, t+k), k = \lambda h^2$, such that*

$$P_h u = Pu + \sum_{j=1}^{m-1} \Gamma_j(x,t;u,\lambda)h^{2j} + R_m(x,t;\xi,\tau,u,\lambda)h^{2m},$$

where the functions $\Gamma_j(x,t;u,\lambda) : \overline{\Omega} \to \mathbb{R}$ are defined by

$$\Gamma_j(x,t;u,\lambda) = \frac{1}{(j+1)!} D_t^{j+1} u(x,t)\lambda^j - \frac{2}{(2j+2)!} D_x^{2j+2} u(x,t)$$

and the remainder term $R_m(x,t;\xi,\tau,u,\lambda) : \overline{\Omega} \to \mathbb{R}$ is

$$R_m(x,t;\xi,\tau,u,\lambda) = \frac{1}{(m+1)!} D_t^{m+1} u(x,\tau)\lambda^m - \frac{2}{(2m+2)!} D_t^{2m+2} u(\xi,t).$$

Proof. We use the fact that u is $m+1$ times differentiable with respect to t and $2m+2$ times *continuously* differentiable with respect to x.

4.2 Remark. For easy reference we write out the first three terms of the error expansion

$$\begin{aligned}
\Gamma_1(x,t;u,\lambda) &= \frac{1}{2} u_{tt}(x,t)\lambda - \frac{1}{12} u_{4x}(x,t) \\
\Gamma_2(x,t;u,\lambda) &= \frac{1}{6} u_{ttt}(x,t)\lambda^2 - \frac{1}{360} u_{6x}(x,t) \\
\Gamma_3(x,t;u,\lambda) &= \frac{1}{24} u_{4t}(x,t)\lambda^3 - \frac{1}{20160} u_{8x}(x,t).
\end{aligned}$$

4.3 Remark. It is possible to provide a priori estimates for all remainder terms. As an example we consider the simplest case of $m = 2$. We must consider a slightly larger strip S'_T with $T' = T + k_0$, $k_0 > 0$. By the triangle inequality

$$|R_2(x,t;\xi,\tau;u,\lambda)| \le \frac{1}{6} \|u_{ttt}\|_{T',\infty} \lambda^2 + \frac{1}{360} \|u_{6x}\|_{T',\infty}$$

for all $(x,t) \in S_T$, and $T' = T + k_0$, $0 < \lambda h^2 \le k_0$. The derivatives $u_{ttt} = u_{3t}$ and u_{6x} satisfy equations similar to the original one, namely

$$\begin{aligned}
P(u_{3t}) &= f_{3t}, \quad t > 0 \\
P(u_{6x}) &= f_{6x}, \quad t > 0
\end{aligned}$$

with the initial conditions of

$$\begin{aligned}
u_{3t}(x,0) &= g^{(6)}(x) + f_{4x}(x,0) + f_{txx}(x,0) + f_{tt}(x,0) \\
u_{6x}(x,0) &= g^{(6)}(x).
\end{aligned}$$

We may then apply the smooth maximum principle to them and estimate

$$\begin{aligned}
\|u_{3t}\|_{T',\infty} &\le \sup_{x\in\mathbb{R}} |g^{(6)}(x) + f_{4x}(x,0) + f_{txx}(x,0) + f_{tt}(x,0)| + T'\|f_{3t}\|_{T',\infty} \\
\|u_{6x}\|_{T',\infty} &\le \sup_{x\in\mathbb{R}} |g^{(6)}(x)| + T'\|f_{6x}\|_{T',\infty}.
\end{aligned}$$

We now turn to the construction of the coefficients of the error expansion.

4.4 Lemma. *(Construction of α) There exists a smooth function*

$$\alpha : \overline{\Omega} \to \mathbb{R}$$

such that

$$\forall T > 0 \, \forall h_0 > 0 \, \exists C(u, T, h_0) > 0 \, \forall h \in (0, h_0) \; : \qquad |u - v_h - \alpha h^2| \le C(u, T, h_0) h^4$$

for all $(x, t) \in \Sigma_h \cap S_T$. The function α is the solution of

$$\begin{aligned} P\alpha &= \Gamma_1(u, \lambda), \quad && t > 0 \\ \alpha &= 0, && t = 0. \end{aligned}$$

Proof. It is important to realise that α and $C(u, T, h_0)$ are completely independent of h.

By lemma 4.1

$$P_h u = Pu + \Gamma_1(u, \lambda) h^2 + R_2(u, \lambda) h^4,$$

and by definition $u - v_h \equiv 0$ on the initial line, which is why we consider the equation

$$P\alpha = \Gamma_1(u, \lambda), \quad t > 0.$$

The choice of $\alpha = 0$ for $t = 0$ allows $u - v_h - \alpha h^2 = 0$ for $t = 0$.

This equation has a unique solution $\alpha : \overline{\Omega} \to \mathbb{R}$ which is infinitely differentiable. α and its derivatives are bounded on any strip S_T. Let α_h be the solution of the corresponding discrete problem

$$\begin{aligned} P_h \alpha_h &= \Gamma_1(u, \lambda), \quad (x, t) \in \Sigma_h \\ \alpha_h^0 &= 0 \end{aligned}$$

By lemma 3.18 there exists a constant $C(\alpha, T)$ such that

$$|\alpha(x, t) - \alpha_h(x, t)| \le C(\alpha, T) h^2,$$

for all $(x, t) \in \Sigma_h \cap S_T$. By the triangle inequality

$$\begin{aligned} |u - v_h - \alpha h^2| &\le |u - v_h - \alpha_h h^2| + |(\alpha - \alpha_h) h^2| \\ &\le |u - v_h - \alpha_h h^2| + C(\alpha, T) h^4. \end{aligned}$$

By construction

$$u(x, 0) - v_h(x, 0) - \alpha_h(x, 0) h^2 \equiv 0$$

and

$$\begin{aligned}P_h(u - v_h - \alpha_h h^2) &= P_h u - P_h v_h - (P_h \alpha_h) h^2 \\ &= \big(Pu + \Gamma_1(u,\lambda) h^2 + R_2(u,\lambda) h^4\big) - P_h v_h - (P_h \alpha_h) h^2 \\ &= R_2(u,\lambda) h^4\end{aligned}$$

because $Pu = P_h v_h = f$ and $P_h \alpha_h = \Gamma_1(u, \lambda)$. As shown in remark 4.3 the remainder terms are bounded on every strip S_T. In particular there exists a constant $M_2(u, T, h_0)$, such that

$$|R_2(x, t; \xi, \tau, u, \lambda)| \leq M_2(u, T, h_0)$$

for all $(x, t) \in \Sigma_h \cap S_T$ with $h < h_0$. By the discrete maximum principle

$$|u - v_h - \alpha_h h^2| \leq T M_2(u, T, h_0) h^4$$

and we finally have the desired estimate of

$$|u - v_h - \alpha h^2| \leq C(u, T, h_0) h^4$$

for all $(x, t) \in \Sigma_h \cap S_T$. The constant $C(u, T, h_0)$ does not depend on $h \in (0, h_0)$ but can be estimated in terms of λ, h_0 and the derivatives of g and f.

4.5 Lemma. *(Construction of β) There exists a smooth function*

$$\beta : \overline{\Omega} \to \mathbb{R}$$

such that

$$\forall T > 0\, \forall h_0 > 0\, \exists C(u, T, h_0) > 0\, \forall h \in (0, h_0) : \quad |u - v_h - \alpha h^2 - \beta h^4| \leq C(u, T, h_0) h^6$$

for all $(x, t) \in \Sigma_h \cap S_T$.

Proof. During my search for β I fell into the following trap. For *any* functions β, β_h we have the estimate

$$\begin{aligned}|u - v_h - \alpha h^2 - \beta h^4| \leq\ & |u - v_h - \alpha_h h^2 - \beta_h h^4| \\ & + |(\beta - \beta_h) h^4| + |(\alpha - \alpha_h) h^2|.\end{aligned}$$

By lemma 4.1

$$P_h u = Pu + \Gamma_1(u, \lambda) h^2 + \Gamma_2(u, \lambda) h^4 + R_3(u, \lambda) h^6$$

and I defined β_h as the solution of the difference equation

$$\begin{aligned}P_h \beta_h &= \Gamma_2(u, \lambda), \quad (x, t) \in \Sigma_h \\ \beta_h^0 &= 0\end{aligned}$$

and β as the solution of the corresponding differential equation

$$\begin{aligned} P\beta &= \Gamma_2(u,\lambda), \quad && t > 0 \\ \beta &= 0 && t = 0. \end{aligned}$$

Then certainly $u - v_h - \alpha_h h^2 - \beta_h h^4 = O(h^6)$ and $(\beta - \beta_h)h^4 = O(h^6)$. However $(\alpha - \alpha_h)h^2$ is only $O(h^4)$ which is not good enough !

Instead I discovered that one should proceed in the following fashion. Write

$$P_h u = Pu + \Gamma_1(u,\lambda)h^2 + \Gamma_2(u,\lambda)h^4 + R_3(u,\lambda)h^6$$

and

$$P_h \alpha = P\alpha + \Gamma_1(\alpha,\lambda)h^2 + R_2(\alpha,\lambda)h^4.$$

Then

$$\begin{aligned} P_h(u - v_h - \alpha h^2) = \big(&Pu + \Gamma_1(u,\lambda)h^2 + \Gamma_2(u,\lambda)h^4 + R_3(u,\lambda)h^6\big) \\ &- P_h v_h - \big(P\alpha + \Gamma_1(\alpha,\lambda)h^2 + R_2(\alpha,\lambda)h^4\big)\, h^2 \\ &\qquad = \left[\Gamma_2(u,\lambda) - \Gamma_1(\alpha,\lambda)\right] h^4 + \left[R_3(u,\lambda) - R_2(\alpha,\lambda)\right] h^6 \end{aligned}$$

because $Pu = P_h v_h = f$ and $P\alpha = \Gamma_1(u,\lambda)$ and now the choice of

$$\begin{aligned} P_h \beta_h &= \Gamma_2(u,\lambda) - \Gamma_1(\alpha,\lambda), \quad (x,t) \in \Sigma_h \\ \beta_h^0 &= 0 \end{aligned}$$

becomes the natural one. Then

$$P_h(u - v_h - \alpha h^2 - \beta_h h^4) = \left[R_3(u,\lambda) - R_2(\alpha,\lambda)\right] h^6$$

so that

$$|u - v_h - \alpha h^2 - \beta_h h^4| \le T\left[M_3(u,T,h_0) + M_2(\alpha,T,h_0)\right] h^6$$

on $\Sigma_h \cap S_T$. If β is chosen as the solution of the corresponding differential equation

$$P\beta = \Gamma_2(u,\lambda) - \Gamma_1(\alpha,\lambda), \quad (x,t) \in \overline{\Omega}$$

with $\beta \equiv 0$ on the initial line, we are given the estimate

$$|(\beta - \beta_h)h^4| \le C(\beta,T)h^6, \quad (x,t) \in \Sigma_h \cap S_T$$

which finally allows us the pleasure of estimating

$$\begin{aligned} |u - v_h - \alpha h^2 - \beta h^4| &\le |u - v_h - \alpha h^2 - \beta_h h^4| + |(\beta - \beta_h)h^4| \\ &\le C(u,T,h_0)h^6. \end{aligned}$$

4.6 Lemma. *(Construction of γ) There exists a smooth function*

$$\gamma : \overline{\Omega} \to \mathbb{R}$$

such that

$$\forall T > 0\, \forall\, h_0 > 0\, \exists\, C(u,T,h_0) > 0\, \forall\, h \in (0,h_0)\ : \qquad |u - v_h - \alpha h^2 - \beta h^4 - \gamma h^6| \le C(u,T,h_0)h^8$$

for all $(x,t) \in \Sigma_h \cap S_T$.

Inspired by the proof of the previous lemmata we write

$$\begin{aligned} P_h u &= Pu + \Gamma_1(u,\lambda)h^2 + \Gamma_2(u,\lambda)h^4 + \Gamma_3(u,\lambda)h^6 + R_4(u,\lambda)h^8 \\ P_h\alpha &= P\alpha + \Gamma_1(\alpha,\lambda)h^2 + \Gamma_2(\alpha,\lambda)h^4 + R_3(\alpha,\lambda)h^6 \\ P_h\beta &= P\beta + \Gamma_1(\beta,\lambda)h^2 + R_2(\beta,\lambda)h^4 \end{aligned}$$

from which we deduce that

$$\begin{aligned} P_h(u - v_h - \alpha h^2 - \beta h^4) = {} & [\Gamma_3(u,\lambda) - \Gamma_2(\alpha,\lambda) - \Gamma_1(\beta,\lambda)]\, h^6 \\ & + [R_4(u,\lambda) - R_3(\alpha,\lambda) - R_2(\beta,\lambda)]\, h^8, \end{aligned}$$

leaving us with the natural choice of

$$\begin{aligned} P_h\gamma_h &= \Gamma_3(u,\lambda) - \Gamma_2(\alpha,\lambda) - \Gamma_1(\beta,\lambda), \quad (x,t) \in \Sigma_h \cap S_T \\ \gamma_h^0 &= 0. \end{aligned}$$

Let γ be the solution of the corresponding differential equation, then as in the previous proofs

$$|u - v_h - \alpha h^2 - \beta h^4 - \gamma h^6| \le C(u,T,h_0)h^8$$

because there exists a constant $C(\gamma,T)$ such that

$$|(\gamma - \gamma_h)h^6| \le C(\gamma,T)h^8$$

for all $(x,t) \in \Sigma_h \cap S_T$ and

$$\begin{aligned} & |P_h(u - v_h - \alpha h^2 - \beta h^4 - \gamma_h h^6)| \\ & \qquad \le [M_4(u,T,h_0) + M_3(\alpha,T,h_0) + M_2(\beta,T,h_0)]\, h^8 \end{aligned}$$

combined with

$$u - v_h - \alpha h^2 - \beta h^4 - \gamma_h h^6 = 0$$

on the initial line of $t = 0$, yields

$$\begin{aligned} & |u - v_h - \alpha h^2 - \beta h^4 - \gamma_h h^6| \\ & \qquad \le T\, [M_4(u,T,h_0) + M_3(\alpha,T,h_0) + M_2(\beta,T,h_0)]\, h^8 \end{aligned}$$

This completes the construction of γ.

Obviously we may continue the construction of coefficients α, β, γ indefinitely. Rewriting $\alpha = \alpha_1, \beta = \alpha_2$ and $\gamma = \alpha_3$ we find

$$\begin{aligned} P\alpha_1 &= \Gamma_1(u,\lambda) \\ P\alpha_2 &= \Gamma_2(u,\lambda) - \Gamma_1(\alpha_1,\lambda) \\ P\alpha_3 &= \Gamma_3(u,\lambda) - \Gamma_2(\alpha_1,\lambda) - \Gamma_1(\alpha_2,\lambda) \end{aligned}$$

with the initial conditions of $\alpha_j(x,0) \equiv 0$.

We have the following main theorem.

4.7 Theorem. *Let $\alpha_j : \overline{\Omega} \to \mathbb{R}$ be the solutions of the recursively defined differential equations*

$$\begin{aligned} P\alpha_j &= \Gamma_j(u,\lambda) - \sum_{i=1}^{j-1} \Gamma_{j-i}(\alpha_i,\lambda), \quad && t > 0 \\ \alpha_j &= 0 && t = 0 \end{aligned}$$

then

$$\forall T > 0 \, \forall m \, \forall h_0 > 0 \, \exists C(u,T,m,h_0) > 0 \, \forall h \in (0,h_0) :$$
$$\left| u - v_h - \sum_{j=1}^{m} \alpha_j h^{2j} \right| \leq C(u,T,m,h_0) h^{2m+2}$$

for all $(x,t) \in \Sigma_h \cap S_T$.

Proof. The existence of the α_j is established by induction on j. The crucial point is that the $(j+1)$'st inhomogeneous term is smooth with derivatives that are bounded on every strip $0 \leq t \leq T$, provided that $u, \alpha_1, \alpha_2, \ldots \alpha_j$ are smooth with derivatives that are bounded on every strip. We may then apply our existence theorem and advance to α_{j+1} which will be smooth with derivatives that are bounded on every strip.

The main part of the theorem follows the pattern that has been firmly established in the previous lemmata.

We write

$$\begin{aligned} P_h u &= Pu + \sum_{\nu=1}^{m+1} \Gamma_\nu(u,\lambda) h^{2\nu} + R_{m+2}(u,\lambda) h^{2m+4} \\ P_h \alpha_1 &= P\alpha_1 + \sum_{\mu=1}^{m} \Gamma_\mu(\alpha_1,\lambda) h^{2\mu} + R_{m+1}(\alpha_1,\lambda) h^{2m+2} \\ P_h \alpha_2 &= P\alpha_2 + \sum_{\mu=1}^{m-1} \Gamma_\mu(\alpha_2,\lambda) h^{2\mu} + R_m(\alpha_2,\lambda) h^{2m} \end{aligned}$$

and in general

$$P_h\alpha_j = P\alpha_j + \sum_{\mu=1}^{m+1-j} \Gamma_\mu(\alpha_j,\lambda)h^{2\mu} + R_{m+2-j}(\alpha_j,\lambda)h^{2(m+2-j)}$$

for $j = 1, 2, \ldots, m$. Then

$$P_h u - P_h v_h - \sum_{j=1}^{m}(P_h\alpha_j)h^{2j} = \left[R_{m+2}(u,\lambda) - \sum_{j=1}^{m} R_{m+2-j}(\alpha_j,\lambda)\right] h^{2(m+2)}$$
$$+ \sum_{\nu=1}^{m+1}\Gamma_\nu(u,\lambda)h^{2\nu} - \sum_{j=1}^{m}\left(\sum_{\mu=1}^{m+1-j}\Gamma_\mu(\alpha_j,\lambda)h^{2\mu}\right)h^{2j} - \sum_{j=1}^{m}(P\alpha_j)h^{2j} \quad (4.1)$$

The double sum

$$\sum_{j=1}^{m}\left(\sum_{\mu=1}^{m+1-j}\Gamma_\mu(\alpha_j,\lambda)h^{2\mu}\right)h^{2j}$$

is calculated by summing over constant $\nu = j + \mu$, which yields

$$\sum_{\nu=2}^{m+1}\sum_{j=1}^{\nu-1}\Gamma_{\nu-j}(\alpha_j,\lambda)h^{2\nu}.$$

Thus we discover that

$$\sum_{\nu=1}^{m+1}\Gamma_\nu(u,\lambda)h^{2\nu} - \sum_{j=1}^{m}\left(\sum_{\mu=1}^{m+1-j}\Gamma_\mu(\alpha_j,\lambda)h^{2\mu}\right)h^{2j}$$
$$= \sum_{\nu=1}^{m+1}\Gamma_\nu(u,\lambda)h^{2\nu} - \sum_{\nu=2}^{m+1}\sum_{j=1}^{\nu-1}\Gamma_{\nu-j}(\alpha_j,\lambda)h^{2\nu}$$
$$= \underbrace{\Gamma_1(u,\lambda)}_{P\alpha_1}h^2 + \sum_{\nu=2}^{m}\underbrace{\left(\Gamma_\nu(u,\lambda) - \sum_{j=1}^{\nu-1}\Gamma_{\nu-j}(\alpha_j,\lambda)\right)}_{P\alpha_\nu}h^{2\nu}$$
$$+ \left(\Gamma_{m+1}(u,\lambda) - \sum_{j=1}^{m}\Gamma_{m+1-j}(\alpha_j,\lambda)\right)h^{2(m+1)}$$

and it is now apparent that equation (4.1) reduces to

$$P_h u - P_h v_h - \sum_{j=1}^{m}(P_h\alpha_j)h^{2j} = \left(\Gamma_{m+1}(u,\lambda) - \sum_{j=1}^{m}\Gamma_{m+1-j}(\alpha_j,\lambda)\right)h^{2(m+1)}$$
$$+ \left[R_{m+2}(u,\lambda) - \sum_{j=1}^{m}R_{m+2-j}(\alpha_j,\lambda)\right]h^{2(m+2)}$$

The choice of $P_h\alpha_{m+1,h}$ and α_{m+1} as well as the rest of the proof are now obvious.

4.8 Example. We continue with our basic example of $u(x,t) = e^{-t}\sin(x)$ and $v_h(x,t) = \cos(h)^n \sin(x)$ where $t = nk, k = \lambda h^2, \lambda = \frac{1}{2}$. Then

$$P\alpha = \Gamma_1\left(u, \frac{1}{2}\right) = \frac{1}{6}u$$

and $\alpha(x,t) = \frac{1}{6}te^{-t}\sin(x)$ is easily seen to be the unique solution with $\alpha(x,0) = 0$. Since

$$P\beta = \Gamma_2\left(u, \frac{1}{2}\right) - \Gamma_1\left(\alpha, \frac{1}{2}\right)$$

we calculate

$$\Gamma_2\left(u, \frac{1}{2}\right) = \frac{1}{24}u_{ttt} - \frac{1}{360}u_{6x} = -\frac{1}{24}u + \frac{1}{360}u = -\frac{14}{360}u = -\frac{7}{180}u$$

and

$$\begin{aligned}\Gamma_1\left(\alpha, \frac{1}{2}\right) &= \frac{1}{4}\alpha_{tt} - \frac{1}{12}\alpha_{4x} = \frac{1}{4}\left(\frac{1}{6}u + \alpha_{xx}\right)_t - \frac{1}{12}\alpha \\ &= \frac{1}{24}u_t + \frac{1}{4}\left(\frac{1}{6}u + \alpha_{xx}\right)_{xx} - \frac{1}{12}\alpha = \frac{1}{24}u_t + \frac{1}{24}u_{xx} + \frac{1}{4}\alpha_{4x} - \frac{1}{12}\alpha \\ &= -\frac{1}{12}u + \frac{1}{6}\alpha\end{aligned}$$

so that

$$P\beta = -\frac{7}{180}u + \frac{1}{12}u - \frac{1}{6}\alpha = \frac{8}{180}u - \frac{1}{36}tu = \frac{1}{180}(8-5t)e^{-t}\sin(x)$$

with $\beta(x,0) = 0$. The solution is $\beta = \frac{1}{180}\left(8t - \frac{5}{2}t^2\right)e^{-t}\sin(x)$, because

$$\beta_t = \frac{1}{180}(8-5t)e^{-t}\sin(x) - \beta$$

while $\beta_{xx} = -\beta$, thus

$$P\beta = \frac{1}{180}(8-5t)e^{-t}\sin(x)$$

and $\beta(x,0) = 0$. The claim of theorem 4.7 is merely that

$$\frac{u - v_h - \alpha h^2 - \beta h^4}{h^6}$$

is bounded on every strip S_T for small h.

4.9 Remark. At present there is no guarantee that the sequence of constants

$$\{C(u, T, m, h_0)\}_{m=0}^{\infty}$$

is convergent or even bounded independent of m. My best guess is that it blows up quite rapidly as m tends to infinity. I do not in any way claim that the infinite series

$$\sum_{j=1}^{\infty} \alpha_j h^{2j}$$

is convergent.

Theorem 4.7 does have applications in practical error estimation. The basic result of $|u - v_h| \leq C(u, T)h^2$ is not very useful, in the sense that the constant $C(u, T)$ depends on the exact solution u which we are trying to compute and sufficient information may not be available to estimate C. Instead we fix λ and compute v_h, v_{2h}, then by theorem 4.7

$$u(x, t) - v_h(x, t) = \alpha(x, t)h^2 + \beta(x, t)h^4 + \gamma(x, t)h^6 + O(h^8)$$

for all $(x, t) \in \Sigma_h \cap S_T$ for some particular value of T and

$$u(x, t) - v_{2h}(x, t) = 4\alpha(x, t)h^2 + 16\beta(x, t)h^4 + 64\gamma(x, t)h^6 + O(h^8)$$

for all $(x, t) \in \Sigma_{2h} \cap S_T$ so that

$$v_h - v_{2h} = 3\alpha h^2 + 15\beta h^4 + 63\gamma h^6 + O(h^8)$$

for all $(x, t) \in \Sigma_{2h} \cap S_T$. Then the quantity $\frac{1}{3}(v_h - v_{2h})$ is essentially the second order error term of αh^2, since

$$\frac{1}{3}(v_h - v_{2h}) = \alpha h^2 + 5\beta h^4 + 21\gamma h^6 + O(h^8).$$

We may therefore use the values of $\frac{1}{3}(v_h - v_{2h})$ as error estimates or add them to our values of v_h in order to improve accuracy,

$$u - v_h - \frac{1}{3}(v_h - v_{2h}) = -4\beta h^4 - 20\gamma h^6 + O(h^8),$$

at the expense of an error estimate. This is a standard technique called Richardson extrapolation.

Theorem 4.7 may be verified experimentally. We find that

$$u - v_{4h} = 16\alpha h^2 + 256\beta h^4 + 4096\gamma h^6 + O(h^8)$$

so that

$$v_{2h} - v_{4h} = 12\alpha h^2 + 240\beta h^4 + 4032\gamma h^6 + O(h^8)$$

and the fraction

$$\frac{v_{2h}-v_{4h}}{v_h-v_{2h}} = \frac{12\alpha h^2+240\beta h^4+4032\gamma h^6+O(h^8)}{3\alpha h^2+15\beta h^4+63\gamma h^6+O(h^8)}$$
$$= \frac{4+80\frac{\beta}{\alpha}h^2+1344\frac{\gamma}{\alpha}h^4+O(h^6)}{1+5\frac{\beta}{\alpha}h^2+21\frac{\gamma}{\alpha}h^4+O(h^6)}$$

will tend to 4 when h tends to zero provided that $\alpha \neq 0$.

4.10 Remark. Originally my supervisor Ole Østerby presented me with substantial experimental evidence of this effect and demonstrated the use of the error estimate $\frac{1}{3}(v_h - v_{2h})$ in practical applications [3] and I decided to try to establish the basic error expansion of

$$u - v_h = ah + bk + ch^2 + dhk + ek^2 + \ldots$$

and after much hardship I finally came upon theorem 4.7. I would dearly like to claim credit for this small piece of mathematics, but I seriously doubt that it has been overlooked by everybody else.

4.11 Remark. The work of this chapter extends immediately to the more general choice of

$$Pu = u_t - au_{xx} - bu_x - cu = f$$

as the key ingredients of smoothness and maximum principles are available in this case. It is much more difficult to extend the results to initial-boundary value problems because of the difficulty of providing global estimates for the remainder terms. It is however easy in the very special case of the heat equation on a bounded rectangle with smooth Dirichlet boundary conditions.

Chapter 5

Maximum principles

In this chapter we develop maximum principles for equations of the type

$$u_t - au_{xx} - bu_x - cu = f\ .$$

We show that any solution is essentially bounded in terms of the driving force f and the initial-boundary values. Inspired by these maximum principles we will derive discrete analogies and use those in our construction of smooth solutions. We only consider sets Ω of the type

$$\Omega = \{(t,x)|0 < t < T,\ \phi_1(t) < x < \phi_2(t)\},$$

generated by a pair of continuous functions $\phi_i : [0,T] \to \mathbb{R}$ with $\phi_1(t) < \phi_2(t)$ for all $t \in [0,T]$. We divide the boundary of Ω into two disjoint sets

$$\begin{aligned}\delta''\Omega &= \{(t,x)\,|\,t = T \wedge \phi_1(t) < x < \phi_2(t)\}\\ \delta'\Omega &= \overline{\Omega} - \delta''\Omega.\end{aligned}$$

We shall refer to $\delta'\Omega$ as the *parabolic boundary* of Ω. It is precisely on this set that one would normally prescribe values of u. We shall refer to $\delta''\Omega$ as the *final* line.

5.1 Lemma. *If $u_t - u_{xx} < 0$ then the maximum of u is attained only on the parabolic boundary,*

$$u(t,x) < \max_{\delta'\Omega} u,$$

for all $(t,x) \in \overline{\Omega} - \delta'\Omega$.

Proof. The continuous function u does indeed attain its maximum in some point (t,x) in the compact set $\overline{\Omega}$. We must eliminate the possibility of $u \in \Omega$ or $u \in \delta\Omega''$.
Assume that $(t,x) \in \Omega$. Then (t,x) is an internal maximum point and by Taylor's formula we must have $u_t = 0$ and $u_{xx} \leq 0$. Clearly this is incompatible

with our basic assumption of $u_{xx} > u_t$.
If $(t, x) \in \delta''\Omega(t = T)$ then $u_t \geq 0$, because otherwise there would be points in Ω with even greater values of u, as

$$u(T - k, x) = u(T, x) - u_t(T, x)k + O(k^2).$$

Further, the value of $u(T, x)$ is a maximum for the restriction of u to the line $t = T$, therefore $u_{xx} \leq 0$. But this violates our assumption $u_{xx} > u_t$. Therefore $(t, x) \in \delta'\Omega$ and we have proved the lemma.

5.2 Theorem. *If $u_t - u_{xx} \leq 0$ then the maximum of u is attained on the parabolic boundary $\delta'\Omega$,*

$$\max_{\overline{\Omega}} u = \max_{\delta'\Omega} u.$$

Proof. We define the auxiliary function v by

$$v(t, x) = u(t, x) - \varepsilon t,$$

with $\epsilon > 0$. By construction $v_t - v_{xx} = (u_t - \varepsilon) - u_{xx} < 0$ and by the previous lemma

$$v(t, x) < \max_{\delta'\Omega} v,$$

for all $(t, x) \in \overline{\Omega} - \delta'\Omega$. In particular

$$\max_{\overline{\Omega}} v = \max_{\delta'\Omega} v.$$

Now

$$\max_{\overline{\Omega}} u = \max_{\overline{\Omega}} \ (v + \varepsilon t) \leq \max_{\overline{\Omega}} v + \varepsilon T \leq \max_{\delta'\Omega} u + \varepsilon T.$$

The proof is completed by letting ε tend to zero.
In the case of $u_t - u_{xx} = 0$ we may apply the theorem to u as well as $-u$ yielding

5.3 Corollary. *If $u_t - u_{xx} = 0$, then u is bounded by its boundary values in the sense that*

$$\max_{\overline{\Omega}} |u| = \max_{\delta'\Omega} |u|.$$

We move on to the following case of

5.4 Lemma. *If $c < 0$ and $u_t = au_{xx} + bu_x + cu$, with $a \geq 0$ then either $u \equiv 0$ or the maximum of $|u|$ is attained only on the $\delta'\Omega$.*

$$|u(t, x)| < \max_{\overline{\Omega}} |u|,$$

for all $(t, x) \in \overline{\Omega} - \delta'\Omega$.

Proof. The proof is by Taylor analysis of u. Assume that u has a local maximum in $(t, x) \in \Omega$. At this point we have $u_t = u_x = 0$ and as usual $u_{xx} \leq 0$. The differential equation reduces to

$$0 = \underbrace{au_{xx}}_{\leq 0} + 0 + cu \Rightarrow cu \geq 0$$

and we are forced to conclude that $u \leq 0$, because $c < 0$. If u has a local maximum in $(t, x) \in \delta''\Omega$, we still have $u_x = 0$ and $u_{xx} \leq 0$, but $u_t \geq 0$. The differential equation supplies the inequality

$$0 \leq u_t = \underbrace{au_{xx}}_{\leq 0} + 0 + cu \Rightarrow cu \geq 0$$

and our conclusion $u \leq 0$ remains in force. By applying this analysis to $-u$ we deduce that if u has a local minimum somewhere in $\overline{\Omega} - \delta'\Omega$, then this minimum value must be non-negative.
Now assume that some point $(t, x) \in \overline{\Omega} - \delta'\Omega$, actually maximizes the value of $|u|$, that is

$$|u(t, x)| = \max_{\overline{\Omega}} |u|.$$

There are two distinct possibilities either $u(t, x) = \max_{\overline{\Omega}} u$ or $u(t, x) = \min_{\overline{\Omega}} u$. We treat only the first case as the second case is quite similar. If the point (t, x) maximizes u then by our previous analysis $u(t, x) \leq 0$. The *minimum* value of u is assumed at some point (t', x') in $\overline{\Omega}$ and

$$\min_{\overline{\Omega}} u = u(t', x') \leq u(t, x) \leq 0.$$

In particular this means, that $|u(t', x')| \geq |u(t, x)|$. Clearly the case of

$$|u(t', x')| > |u(t, x)| = \max_{\overline{\Omega}} |u|,$$

is outright impossible and we are left to conclude that not only is $|u(t', x')| = |u(t, x)|$, but

$$\min_{\overline{\Omega}} u = u(t', x') = u(t, x) = \max_{\overline{\Omega}} u$$

and u is indeed a constant. The only such solution admitted by the differential equation is the trivial one, because $c \neq 0$.
Summary: If $u \not\equiv 0$ we can not possibly find a point $(t, x) \in \overline{\Omega} - \delta'\Omega$ that maximizes $|u|$ and we are forced to conclude that $|u|$ achieves its maximum value only on the parabolic boundary, that is

$$|u(t, x)| < \max_{\delta'\Omega} |u|,$$

for all $(t, x) \in \overline{\Omega} - \delta'\Omega$.

The case of $c \leq 0$ follows as an immediate corollary to this theorem.

5.5 Corollary. *If* $c \leq 0$ *and* $u_t = au_{xx} + bu_x + cu$ *then*

$$\max_{\overline{\Omega}} |u| = \max_{\delta'\Omega} |u|$$

Proof. We introduce an auxiliary function v defined by

$$v(t,x) = u(t,x)e^{-\epsilon t}.$$

The previous theorem applies to v because

$$v_t = (u_t - \epsilon u)\, e^{-\epsilon t} = (au_{xx} + bu_x + cu - \epsilon u)\, e^{-\epsilon t} = av_{xx} + bv_x + \underbrace{(c-\epsilon)}_{<0} v,$$

so that either $v \equiv 0$ or

$$|v(t,x)| < \max_{\overline{\Omega}} |v|,$$

for all $(t,x) \in \overline{\Omega} - \delta'\Omega$. Now $v \equiv 0$ iff $u \equiv 0$ in which case the theorem is trivially true. Assuming $u \not\equiv 0$ we deduce, that

$$|u(t,x)| = e^{\epsilon t}|v(t,x)| \leq e^{\epsilon t}\max_{\delta'\Omega} |v| \leq e^{\epsilon t}\max_{\delta'\Omega} |u| \leq e^{\epsilon T}\max_{\delta'\Omega} |u|,$$

for all $(t,x) \in \overline{\Omega}$. Letting ϵ tend to zero gives us the desired estimate.

We are now ready to prove the following theorem

5.6 Theorem. *If* $u_t - au_{xx} - bu_x - cu = f$, *then*

$$\max_{\overline{\Omega}} |u| \leq \left(\max_{\delta'\Omega} |u| + T\sup_{\overline{\Omega}} |f|\right) e^{CT},$$

where $C = \max\{0, \max_{\overline{\Omega}} c\}$.

Proof. Assume that $(t,x) \in \overline{\Omega} - \delta'\Omega$ maximizes u. As before $u_t \geq 0, u_x = 0$ and $u_{xx} \leq 0$. The differential equation reduces to

$$0 \leq u_t = au_{xx} + 0 + cu + f.$$

But we can not conclude that $cu \geq 0$ unless $f \leq 0$ and given $cu \geq 0$ we can not conclude $u \leq 0$ unless $c < 0$. Inspired by the proof of the previous theorems we define a pair of auxiliary functions v, w by

$$\begin{aligned} v(t,x) &= u(t,x)e^{-(C+\epsilon)t} \\ w(t,x) &= v(t,x) - tF, \end{aligned}$$

where $F = \max\{0, \max_{\overline{\Omega}} f\}$ dominates f completely. Clearly v is not quite good enough, because

$$\begin{aligned} v_t = (u_t - (C+\epsilon)u)\, e^{-(C+\epsilon)t} &= (au_{xx} + bu_x + cu - (C+\epsilon)u + f)\, e^{-(C+\epsilon)t} \\ &= av_{xx} + bv_x + \underbrace{(c - C - \epsilon)}_{<0} v + \underbrace{fe^{-(C+\epsilon)t}}_{\text{quite unknown}} \end{aligned}$$

but w does the trick

$$\begin{aligned}
w_t &= v_t - F \\
&= av_{xx} + bv_x + (c - C - \epsilon)\, v + fe^{-(C+\epsilon)t} - F \\
&= aw_{xx} + bw_x + (c - C - \epsilon)\,(w + tF) + fe^{-(C+\epsilon)t} - F \\
&= aw_{xx} + bw_x + (c - C - \epsilon)\, w + \underbrace{(c - C - \epsilon)\, tF + fe^{-(C+\epsilon)t} - F}_{\le 0}.
\end{aligned}$$

The continuous function w will assume its maximum value at some point (t_0, x_0) in the compact set $\overline{\Omega}$. There are two possibilities, either $(t_0, x_0) \in \overline{\Omega} - \delta'\Omega$ or $(t_0, x_0) \in \delta'\Omega$.
If $(t_0, x_0) \in \overline{\Omega} - \delta'\Omega$, then $w(t, x) \le w(t_0, x_0) \le 0$ and

$$u(t, x) = v(t, x)e^{(C+\epsilon)t} = (w(t, x) + tF)e^{(C+\epsilon)t} \le tFe^{(C+\epsilon)t}.$$

If $(t_0, x_0) \in \delta'\Omega$ we have

$$\begin{aligned}
u(t, x) &= (w(t, x) + tF)e^{(C+\epsilon)t} \\
&\le (w(t_0, x_0) + tF)e^{(C+\epsilon)t} \\
&\le (u(t_0, x_0) + tF)e^{(C+\epsilon)t} \\
&\le \left(\max_{\delta'\Omega} |u| + TF \right) e^{(C+\epsilon)t}.
\end{aligned}$$

In either case we have the desired estimate

$$u(t, x) \le \left(\max_{\delta'\Omega} |u| + T \sup_{\overline{\Omega}} |f| \right) e^{(C+\epsilon)t}.$$

The function $-u$ solves the same type of equation as u, but with f replaced by $-f$. Applying the previous reasoning to $-u$ we may finally conclude that

$$|u(t, x)| \le \left(\max_{\delta'\Omega} |u| + T \sup_{\overline{\Omega}} |f| \right) e^{(C+\epsilon)t}.$$

As usual we conclude the proof by letting ϵ tend to zero.
The following uniqueness theorem is immediate

5.7 Theorem. *If $u_t = au_{xx} + bu_x + cu$ in Ω and $u \equiv 0$ on $\delta'\Omega$ then $u \equiv 0$ on $\overline{\Omega}$.*

5.8 Remark. The proofs of the first two theorems are identical to the proofs provided in Fritz John's book [1]. I have merely split his proof in two for the sake of clarity. The case of $u_t - au_{xx} - bu_x - cu = 0$ was posed as an exercise with suitable hints in the same text. The generalisation to non-zero f became straightforward once I had grasped the principle.

5.9 Remark. Throughout this chapter we have hardly imposed any conditions on the coefficients of the differential equation. We must have $a \geq 0$, c bounded from above and f bounded. But these are the sole restrictions ! b does not even enter into the problem as it is always annihilated by $u_x = 0$. In the *linear* case these conditions are trivially satisfied as the coefficients are continuous functions on a compact set, but our analysis extends even to the quasi-linear case

$$u_t - a(t,x,u(t,x))u_{xx} - b(t,x,u(t,x))u_x - c(t,x,u(t,x))u \\ = f(t,x,u(t,x)),$$

provided that the function $a : \Omega \times \mathbb{R} \to \mathbb{R}$ is non-negative, the function c is real-valued and bounded from above on $\Omega \times \mathbb{R}$, so that we might choose a finite C and that f is bounded on $\Omega \times \mathbb{R}$. b merely needs to be real-valued on $\Omega \times \mathbb{R}$.

Chapter 6

A Priori Estimates of u_x

In this chapter we limit ourselves to the homogeneous equation

$$u_t = au_{xx} + bu_x + cu$$

and develop estimates of the type

$$\|u_x\|_{K,\infty} = \max_K |u_x| \le C_K \max_{\overline{\Omega}} |u| = C_K \|u\|_\infty$$

for compact subsets K of $\Omega \cup \delta''\Omega$, with C_K independent of u. We shall shamelessly assume that our equations have unique solutions. We shall use the resulting estimates as a source of inspiration when deriving discrete analogues. These will eventually be used to prove the existence of smooth solutions.

6.1 Notation. In this chapter we deal exclusively with differentiable functions. We shall use the notation u_x for the partial derivative of u with respect to x instead of the cumbersome $\frac{\partial^2 u}{\partial x^2}$.

We shall need the following theorem.

6.2 Theorem. *If $u_t < au_{xx} + bu_x + cu$, with $c < 0$ and if $(x,t) \in \overline{\Omega} - \delta'\Omega$ maximizes u locally, then $u(x,t) < 0$. In particular if $u \ge 0$ then no such point can be found and the maximum of u is achieved only on the parabolic boundary $\delta'\Omega$.*

The proof is by Taylor-analysis of u and is similar to the first part of the proof of lemma 5.4.

I have studied Petrowski's [2] treatment of the discrete case with $a = 1, b = c = f = 0$ and performed the trivial extension to the smooth case with a general a and b, $c < 0$ and $f = 0$. As any compact set $K \subset \Omega \cup \delta''\Omega$ can be covered by a finite number of rectangles contained in $\Omega \cup \delta''\Omega$ it suffices to derive estimates of the aforementioned type for rectangles. Petrowski's idea is to study the function

$$\phi(x,t) = P(x,t)u_x(x,t)^2 + Qu(x,t)^2, \quad (x,t) \in R,$$

where $P(x,t) = t(x^2-\alpha^2)^2$ is an ingeniously chosen *cut-off* function for the closed rectangle $R = [-\alpha,\alpha]\times[0,T]$ and Q is a positive constant. As it turns out it is a possible to choose Q_0 so that $\phi_t < a\phi_{xx} + b\phi_x + c\phi$ for all $Q \geq Q_0$ and the maximum of ϕ is attained only on the parabolic boundary where P vanishes by design. Therefore

$$Pu_x^2 \leq \phi \leq \max_{\overline{R}} \phi = \max_{\delta' R} \phi = \max_{\delta' R} Qu^2 = Q\left(\max_{\delta' R}|u|\right)^2$$

and

$$|u_x(x,t)| \leq C_{R'} \max_{\delta' R}|u| \leq C_{R'}\|u\|_\infty,$$

for all (x,t) within any closed rectangle $R' \subset R$ with $\delta' R' \cap \delta' R = \emptyset$. Obviously

$$C_{R'}^2 = Q\left(\inf_{R'} P\right)^{-1}.$$

We now address the question of determining Q_0. We shall need the derivatives ϕ_t, ϕ_x and ϕ_{xx}.

$$\begin{aligned}
\phi &= Pu_x^2 + Qu^2 \\
\phi_x &= P_x u_x^2 + 2Pu_x u_{xx} + 2Quu_x \\
\phi_{xx} &= P_{xx}u_x^2 + 2P_x u_x u_{xx} + 2P_x u_x u_{xx} + 2Pu_{xx}^2 + 2Pu_x u_{xxx} + 2Qu_x^2 + 2Quu_{xx} \\
\phi_t &= P_t u_x^2 + 2Pu_x u_{xt} + 2Quu_t
\end{aligned}$$

Therefore

$$\begin{aligned}
L(\phi) = a\phi_{xx} + b\phi_x + c\phi - \phi_t &= \\
&(aP_{xx} + bP_x + cP - P_t)\,u_x^2 + (2au_{xx} + 2bu_x + cu - 2u_t)\,Qu \\
&\quad + (au_{xxx} + bu_{xx} - u_{xt})\,2Pu_x + 4aP_x u_x u_{xx} + 2Pau_{xx}^2 + 2Qau_x^2.
\end{aligned}$$

We want to choose Q so that this expression is positive ! We examine the terms one by one. The map

$$(x,t) \to aP_{xx} + bP_x + cP - P_t$$

is merely a continuous function on the compact set $\overline{R}$. The term

$$\begin{aligned}
&(2au_{xx} + 2bu_x + cu - 2u_t)\,Qu \\
&\qquad = 2\underbrace{(au_{xx} + bu_x + cu - u_t)}_{0}Qu - cQu^2 = -cQu^2
\end{aligned}$$

is always positive. The term

$$(au_{xxx} + bu_{xx} - u_{xt})$$

must be expanded using the differential equation

$$\begin{aligned}
au_{xxx} + a_x u_{xx} + bu_{xx} + b_x u_x - u_{xt} - a_x u_{xx} - b_x u_x \\
&= (au_{xx} + bu_x - u_t)_x - a_x u_{xx} - b_x u_x \\
&= (au_{xx} + bu_x + cu - u_t)_x - a_x u_{xx} - b_x u_x - (cu)_x \\
&= -a_x u_{xx} - b_x u_x - c_x u - cu_x \\
&= -a_x u_{xx} - (b_x + c)u_x - c_x u.
\end{aligned}$$

In short

$$\begin{aligned}
a\phi_{xx} + b\phi_x + c\phi - \phi_t \\
&= L(P)u_x^2 - cQu^2 - (a_x u_{xx} + (b_x + c)u_x + c_x u)2Pu_x \\
&\quad + 4aP_x u_x u_{xx} + 2Pau_{xx}^2 + 2Qau_x^2
\end{aligned}$$

$$\begin{aligned}
&= L(P)u_x^2 - 2P(b_x + c)u_x^2 + 2Qau_x^2 \\
&\quad - (2Pa_x - 4aP_x)u_x u_{xx} + 2Pau_{xx}^2 \\
&\quad - cQu^2 - 2Pc_x uu_x.
\end{aligned}$$

What remains is to complete the squares. Now

$$\begin{aligned}
2Pa_x - 4aP_x &= 2t(x^2 - \alpha^2)^2 a_x - 16at(x^2 - \alpha^2)x \\
&= 2\left(a_x(x^2 - \alpha^2) - 8ax\right)t(x^2 - \alpha^2),
\end{aligned}$$

so that

$$\begin{aligned}
-(2Pa_x - 4aP_x)u_x u_{xx} + 2Pau_{xx}^2 &= 2a\left(Pu_{xx}^2 - 2\frac{Pa_x - 2aP_x}{2a}\right) \\
&= 2at\left((x^2 - \alpha^2)u_{xx} - \frac{a_x(x^2 - \alpha^2) - 8ax}{2a}u_x\right)^2 \\
&\quad - 2at\left(\frac{a_x(x^2 - \alpha^2) - 8ax}{2a}u_x\right)^2 \geq -t\frac{(a_x(x^2 - \alpha^2) - 8ax)^2}{2a}u_x^2.
\end{aligned}$$

Similarly

$$\begin{aligned}
-cQu^2 - 2Pc_x uu_x &= -cQ\left(u^2 + 2u\frac{Pc_x}{cQ}u_x\right) \\
&= -cQ\left(u^2 + 2u\frac{Pc_x}{cQ}u_x + \left(\frac{Pc_x}{cQ}u_x\right)^2\right) + \frac{(Pc_x)^2}{cQ}u_x^2 \\
&\geq \frac{(Pc_x)^2}{cQ}u_x^2.
\end{aligned}$$

And it finally reduces to

$$\begin{aligned}
&a\phi_{xx} + b\phi_x + c\phi - \phi_t \\
&\geq \Bigg(\underbrace{L(P) - 2P(b_x + c) - t\frac{(a_x(x^2 - \alpha^2) - 8ax)^2}{2a} + \frac{(Pc_x)^2}{cQ}}_{\text{complicated, but continuous function !}}\Bigg) u_x^2 + 2aQu_x^2
\end{aligned}$$

and the existence of Q_0 is almost obvious. Note that $\frac{(Pc_x)^2}{cQ} \geq \frac{(Pc_x)^2}{c}$ for $Q \geq 1$ because $c < 0$. Then we may choose

$$Q_0 = \frac{1}{2\min\limits_R a} \max_R \left(L(P) - 2P(b_x + c) - t\frac{(a_x(x^2 - \alpha^2) - 8ax)^2}{2a} + \frac{(Pc_x)^2}{c} \right)$$

which clearly depends exclusively on R, a, b, c, a_x, b_x, and c_x, but *not* on u. We have proven the following theorem.

6.3 Theorem. *Let* $\sup c < 0$. *Let* $K \subset \Omega \cup \delta''\Omega$ *be a compact set. Then there exists a constant* $C > 0$ *such that*

$$\|u_x\|_{K,\infty} \leq C\|u\|_\infty$$

for all solutions $u : \Omega \cup \delta''\Omega \to \mathbb{R}$ *of the equation*

$$u_t = au_{xx} + bu_x + cu.$$

The constant C *can be calculated in terms of* a, b, c, *and* K.

The restriction $\sup c < 0$ is not severe as the following analysis demonstrates. Define the auxiliary function v by

$$v(x, t) = u(x, t)e^{-(C+\epsilon)t},$$

where $C = \max\{0, \sup c\}$ and $\epsilon > 0$. Then

$$\begin{aligned}
v_t &= (u_t - (C + \epsilon)u)e^{-(C+\epsilon)t} \\
&= (au_{xx} + bu_x + (c - C - \epsilon)u)e^{-(C+\epsilon)t} \\
&= av_{xx} + bv_x + (c - C - \epsilon)v
\end{aligned}$$

and if K is a compact subset of Ω, there exists a constant C_K independent of v, such that

$$\|v_x\|_{K,\infty} \leq C_K\|v\|_\infty.$$

Now $u = ve^{(C+\epsilon)t}$ so that $|u_x| = |v_x|e^{(C+\epsilon)t}$ and

$$\begin{aligned}
\|u_x\|_{K,\infty} &\leq \|v_x\|_{K,\infty}e^{(C+\epsilon)T} \\
&\leq C_K\|v\|_\infty e^{(C+\epsilon)T} \leq C_K\|u\|_\infty e^{(C+\epsilon)T}.
\end{aligned}$$

Letting ϵ tend to zero provides the useful estimate

$$\|u_x\|_{K,\infty} \leq C_K \|u\|_\infty e^{CT}.$$

6.4 Remark. Originally I considered the case of $c < 0$ and $f \leq 0$, as the general case can be reduced to this one through the transformations of chapter 5. I tried and failed to modify Petrowski's function P to cover this case. As we shall demonstrate in chapter 8 there are other ways to eliminates inhomogeneous terms.

Chapter 7

Mixed Problems for the Heat Equation

In this chapter we study the initial-boundary value problems for the differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

We consider sets Ω of the type

$$\Omega = \{(x,t) \in \mathbb{R}^2 \ : \ 0 < t < T, \ \phi_1(t) < x < \phi_2(t)\},$$

where $\phi_1, \phi_2 : [0,T] \to \mathbb{R}$ are continuous functions with

$$\phi_1(t) < \phi_2(t)$$

for $t \in [0,T]$. We divide the boundary $\delta\Omega$ of Ω into two disjoint sets

$$\begin{aligned} \delta''\Omega &= \{(t,x) \,|\, t = T \wedge \phi_1(t) < x < \phi_2(t)\} \\ \delta'\Omega &= \overline{\Omega} - \delta''\Omega. \end{aligned}$$

We shall refer to $\delta'\Omega$ as the *parabolic boundary* of Ω and call $\delta''\Omega$ the *final* line.

Given an initial-boundary condition $g : \delta'\Omega \to \mathbb{R}$ we seek a function

$$u : \Omega \cup \delta''\Omega \to \mathbb{R}$$

such that the differential equation is satisfied at every point of $\Omega \cup \delta''\Omega$ and

$$u(x,t) \to g(p) \quad \text{as} \quad (x,t) \to p \quad \text{with} \quad (x,t) \in \Omega \cup \delta''\Omega$$

for every point $p \in \delta'\Omega$.

7.1 Notation. This chapter is dominated by calculations involving divided differences. In order to simplify the notation I have decided to use v_x instead of

$\delta_x v$. It is much faster to type as well as write by hand and I have gotten quite used to it. Derivatives of smooth functions are only used sporadicly which is why I have chosen to use the cumbersome notation of $\frac{\partial u}{\partial x}$ for those. Remember that $hv_x = h\delta_x v = E_x v - v$.

We shall study the heat equation in a manner quite similar to Petrowski's [2] treatment. In chapter 8 we shall extend these arguments to cover the general case of

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + cu + f.$$

We shall use the method of finite differences and use the standard implicit method for these problems.

Pick $N \in \{1, 2, 3, \dots\}$ at random and set $h = T/N > 0$ and let Σ_h be the grid

$$\Sigma_h = \{(jh, nh) \,:\, j, n \in \mathbb{Z}\}.$$

We are exclusively interested in the grid points $(x, t) \in \overline{\Omega}_h = \Sigma_h \cap \overline{\Omega}$ and we subdivide these into two disjoint sets in the following fashion :

If the adjacent points $(x - h, t), (x, t - h), (x + h, t) \in \overline{\Omega}$ **then** $(x, t) \in \Omega_h$ **else** $(x, t) \in \delta'\Omega_h$.

It is convenient to think of Ω_h as the set of internal nodes and $\delta'\Omega_h$ as the set of parabolic boundary nodes.

On $\delta'\Omega_h$ we define a function $g_h : \delta'\Omega_h \to \mathbb{R}$ in the following fashion. Let $(x, t) \in \delta'\Omega_h$ and let $(x', t') \in \delta'\Omega$ be a point that minimizes the distance between (x, t) and $\delta'\Omega$. Since $\delta'\Omega$ is compact there is a least one such point. Set $g_h(x, t) = g(x', t')$. There might easily be more than one choice for (x', t') in which case we pick one at random. In the limit of $h \to 0$ it will not make any difference.

On the set of internal grid points Ω_h we consider the standard implicit difference equation

$$L(v) = v_{x\overline{x}} - v_{\overline{t}} = \frac{v^n_{j+1} - 2v^n_j + v^n_{j-1}}{h^2} - \frac{v^n_j - v^{n-1}_j}{h} = 0$$

along with the initial-boundary conditions of

$$v = g_h$$

on $\delta'\Omega_h$.

7.2 Remark. Please note that we have chosen $h = k$. It is not really important but it simplifies matters a bit as there is one variable less to keep track of.

We need the following lemmata regarding the difference operator L

7.3 Lemma. *If $L(v) = v_{x\overline{x}} - v_{\overline{t}} > 0$ on Ω_h then*

$$v(x, t) < \max_{\delta'\Omega_h} v$$

for all $(x, t) \in \Omega_h$.

Proof. The inequality $L(v) > 0$ is equivalent to

$$(1+2\lambda)v_j^n < \lambda v_{j+1}^n + \lambda v_{j-1}^n + v_j^{n-1}$$

where $\lambda = \frac{h}{h^2} = h^{-1}$. Let $M = \max_{\overline{\Omega}_h} v$ and assume that M is attained at the point $(jh, nh) \in \Omega_h$. Then $M = v_j^n$ and

$$M = (1+2\lambda)M - 2\lambda M = (1+2\lambda)v_j^n - 2\lambda M$$
$$< \lambda v_{j+1}^n + \lambda v_{j-1}^n + v_j^{n-1} - 2\lambda M \leq M,$$

which is clearly impossible.

7.4 Corollary. *If* $L(v) = v_{x\overline{x}} - v_{\overline{t}} \geq 0$ *on* Ω_h *then*

$$v(x,t) \leq \max_{\delta'\Omega_h} v$$

for all $(x,t) \in \Omega_h$.

Proof. Pick $\epsilon > 0$ at random and define the auxiliary function v_ϵ by

$$v_\epsilon = v - \epsilon t.$$

Then $L(v_\epsilon) = L(v) + \epsilon > 0$ and the lemma 7.3 applies to v_ϵ. Therefore

$$v = v_\epsilon + \epsilon t < \max_{\delta'\Omega_h} v_\epsilon + \epsilon t \leq \max_{\delta'\Omega_h} v + \epsilon T$$

for all points in Ω_h. Letting ϵ tend to zero completes the proof.

7.5 Corollary. *If* $L(v) = 0$ *on* Ω_h *then*

$$\max_{\Omega_h} |v| \leq \max_{\delta'\Omega_h} |v|.$$

Proof. The proof is immediate. Apply the previous corollary to v and $-v$ and obtain the inequality

$$\min_{\delta'\Omega_h} v \leq \min_{\Omega_h} v \leq \max_{\Omega_h} v \leq \max_{\delta'\Omega_h} v$$

which completes the proof.

7.6 Remark. Please take the time to compare these results and the proofs to the smooth results of chapter 5.

Our difference equation is a *finite* system of linear equations. The values of v at the internal grid points are the unknowns and there is an equal number of equations. By corollary 7.5 the homogeneous equation

$$L(v) = 0 \quad \text{in} \quad \Omega_h$$
$$v = 0 \quad \text{on} \quad \delta'\Omega_h$$

has only the trivial solution. Thus our difference equation has precisely one solution for each choice of g_h and we have the following estimate of the solution

$$\max_{\overline{\Omega}_h}|v| \leq \|g\|_\infty$$

which is independent of the grid.

Pick $N \in \{1, 2, 3, \dots\}$ at random and set $h_0 = T/N$ and $h_\nu = 2^{-\nu}h_0$. Set $\Sigma_\nu = \Sigma_{h_\nu}, \Omega_\nu = \Omega_{h_\nu}, \delta'\Omega_\nu = \delta'\Omega_{h_\nu}, g_\nu = g_{h_\nu}$ and let $v^\nu : \overline{\Omega}_\nu \to \mathbb{R}$ be *the* solution of the difference equation

$$\begin{aligned} L(v^\nu) = 0 \quad &\text{in} \quad \Omega_\nu \\ v^\nu = g \quad &\text{on} \quad \delta'\Omega_\nu. \end{aligned}$$

We shall use the grid functions v^ν to generate a solution to our differential equation.

We need to bound the divided differences of v^ν independently of the grid. Although $L(v^\nu) = 0$ implies $L(v^\nu_x) = 0$ we can not use this information directly because we have no a priori estimate for the *boundary* values of v^ν_x. Instead we shall bound v^ν_x independently of ν on every compact subset K of $\Omega \cup \delta''\Omega = \overline{\Omega} - \delta'\Omega$.

We shall need a couple of lemmata that will allow us to cover compact subsets with rectangles that have grid points as corners. We begin with the definition of a μ-rectangle.

7.7 Definition. *A closed rectangle $R = [x_1, x_2] \times [t_1, t_2]$ is called a closed μ-rectangle if the corners*

$$(x_1, t_1), \quad (x_1, t_2), \quad (x_2, t_1), \quad (x_2, t_2)$$

are members of Σ_μ. In general we say that a rectangle R is a μ-rectangle regardless of it topological status if the four corners are members of Σ_μ.

The following lemma is an immediate consequence of the definition and the construction of Σ_μ.

7.8 Lemma. *If R is a μ-rectangle then R is a ν-rectangle for every $\nu \geq \mu$.*

7.9 Lemma. *Let $K \subset \mathbb{R}^2$ be a compact subset such that $K \cap \delta'\Omega = \emptyset$. Then there exists a μ and a finite number n of closed μ-rectangles R_i such that*

$$K \subset \cup_{i=1}^n R_i^\circ$$

and $R_i \cap \delta'\Omega = \emptyset$. The set R_i° is the interior of R_i.

7.10 Remark. Lemma 7.9 is valid for general $K \subset \mathbb{R}^2$. Our interest is however limited to $K \subset \Omega \cup \delta''\Omega$. The following proof is valid for general K.

Proof. We begin by constructing a cover of K. Let $\rho = \text{dist}(K, \delta'\Omega)$. Then $\rho > 0$ by compactness. Let $p = (x,t) \in K$. Consider the ball $B(p,\rho)$. Our goal is to construct a closed μ-rectangle R within this ball with $p \in R^\circ$. Recall that Σ is dense in $\mathbb{R}^2$ so we can find $p' = (x', t') \in \Sigma$ such that

$$\|p - p'\| < \frac{1}{4}\rho.$$

By the definition of Σ p' belongs to Σ_μ for some value of μ. Consider the ball $B\left(p', \frac{3}{4}\rho\right)$. By the triangle inequality $B\left(p', \frac{3}{4}\rho\right) \subset B(p,\rho)$. We now make the following claim :

It is possible to pick $\nu \geq \mu$ and a positive integer m such that

$$\frac{1}{4}\rho < mh_\nu \quad \text{and} \quad \sqrt{2}mh_\nu < \frac{3}{4}\rho$$

The consequences are quite benign:

$$p \in B\left(p', \frac{1}{4}\rho\right) \subset [x' - mh_\nu, x' + mh_\nu] \times [t' - mh_\nu, t + mh_\nu]$$
$$\subset B\left(p', \frac{3}{4}\rho\right) \subset B(p,\rho)$$

and since $B(p,\rho) \cap \delta'\Omega = \emptyset$ we would nearly be done with the proof.

Proof of claim: We need to find ν such that

$$\frac{1}{4}\rho < mh_\nu < \frac{3}{4\sqrt{2}}\rho.$$

Obviously this is not entirely impossible since $\frac{1}{4} < \frac{3}{4\sqrt{2}}$. Set $a = \frac{1}{4}$ and $b = \frac{3}{4\sqrt{2}}$. Pick $\nu \geq \mu$ such that $h_\nu < (b-a)/2$. Let $\left\lfloor \frac{a}{h_\nu} \right\rfloor$ be the greatest integer less than or equal to $\frac{a}{h_\nu}$. Then

$$a - h_\nu < \left\lfloor \frac{a}{h_\nu} \right\rfloor h_\nu \leq a$$

such that

$$a < \left\lfloor \frac{a}{h_\nu} \right\rfloor h_\nu + h_\nu \leq a + h_\nu < a + \frac{b-a}{2} < b$$

and we are done by setting $m = \left\lfloor \frac{a}{h_\nu} \right\rfloor + 1$.

Now let R_x be a closed μ-rectangle such that $x \in R_x^\circ$ and $R_x \cap \delta'\Omega = \emptyset$. Then obviously

$$K \subset \cup_{x \in K} R_x^\circ$$

and by compactness we can cover K with a finite number of those. By lemma 7.8 they are all ν-rectangles for some large value of ν.

We turn to the question of estimating v_x relative to v on a compact set

$$K \subset \Omega \cup \delta''\Omega = \overline{\Omega} - \delta'\Omega.$$

By the previous lemma 7.9 there exists a μ and a finite number n of closed μ rectangles $R_i, i = 1, 2, \ldots, n$ such that

$$K \subset \cup_{i=1}^{n} R_i^{\circ}.$$

By construction some of the rectangles R_i might easily extend beyond the final line of $t = T$ in which case we replace them with

$$R_i' = R_i \cap \{(x, t) \in \mathbb{R}^2 \ : \ t \leq T\}$$

so that

$$K \subseteq \cup_{i=1}^{n} R_i \subset \overline{\Omega} - \delta'\Omega.$$

7.11 Remark. Please recall that by construction the final line of $t = T$ is a grid line so that the R_i' are in fact μ-rectangles.

Now since every compact set $K \subset \overline{\Omega} - \delta'\Omega$ can be covered by a finite number of closed μ-rectangles $R_i \subset \overline{\Omega} - \delta'\Omega$ it suffices to consider such rectangles.

We may assume that our closed μ-rectangle is bounded by the lines $t = 0, t = t_1$ and $x = \pm\alpha$. Consider the Petrowskian $z^\nu : \Omega_\nu \to \mathbb{R}$ given by

$$z^\nu = (v_x^\nu)^2 F + C w^\nu$$

where F is an ingeniously chosen cut-off function for the rectangle given by

$$F(x, t) = t(x^2 - \alpha^2)^2$$

and

$$w^\nu(x, t) = v^\nu(x - h_\nu, t)^2 + v^\nu(x, t - h_\nu)^2 + v^\nu(x + h_\nu, t)^2$$

and $C > 0$ is a real number.

We have the following lemma on z^ν

7.12 Lemma.

$$\exists\, C_0\, \forall\, \nu \geq \mu \ : L(v^\nu) \equiv 0 \; on\, (R - \delta' R) \cap \Omega_\nu$$
$$\Rightarrow \forall\, C \ \geq C_0 \ : \ L(z^\nu) \geq 0 \; on\, (R - \delta' R) \cap \Omega_\nu$$

7.13 Remark. Please note that the curious set of $(R - \delta' R) \cap \Omega_\nu$ merely constitutes the set of internal grid points of R.

The proof of lemma 7.12 is a bit technical and will be postponed until we have studied the consequences. We have the following corollary

7.14 Theorem. *Let R_1 and R_2 be closed μ-rectangles with*

$$R_1 \subseteq R_2 \subset \Omega \cup \delta''\Omega$$

but with $\delta' R_1 \cap \delta' R_2 = \emptyset$. Then there exists a constant $C > 0$ such that

$$\forall\, \nu \geq \mu \;:\; \left(L(v^\nu) \equiv 0 \text{ on } \Omega_\nu \text{ and } v^\nu = g_\nu \text{ on } \delta'\Omega_\nu \Rightarrow \max_{R_1} |v_x^\nu| \leq C\, \|g\|_\infty \right)$$

Proof. If $L(z^\nu) \geq 0$ on all the internal nodes of R_2 then by corollary 7.4 the maximum of z^ν is achieved on a parabolic boundary node of R_2 where F vanishes by design. Thus

$$(v_x^\nu)^2 F \leq z^\nu \leq \max_{\delta' R_2} z^\nu = C \max_{\delta' R_2} w^\nu.$$

On the rectangle R_1 we find

$$(v_x^\nu)^2 \left(\inf_{R_1} F \right) \leq (v_x^\nu)^2 F \leq C \max_{\delta' R_2} w^\nu$$

and since $\delta' R_1 \cap \delta' R_2 = \emptyset$ implies that $\inf_{R_1} F > 0$ we are able to deduce that

$$(v_x^\nu)^2 \leq C \left(\inf_{R_1} F \right)^{-1} \max_{\delta' R_2} w^\nu$$

at all grid points of R_1. By the definition of w^ν and corollary 7.5

$$w^\nu \leq 3 \max_{\overline{\Omega}_\nu} |v^\nu|^2 \leq 3 \|g\|_\infty^2$$

and thus

$$\max_{R_1} |v_x^\nu| \leq \sqrt{3\, C \left(\inf_{R_1} F \right)^{-1}}\, \|g\|_\infty$$

and we are done with the proof of this corollary.

We now turn to the proof of lemma 7.12. We will suppress the references to ν for the sake of notational simplicity as well as generality. We need the following lemma.

7.15 Lemma. *If $L(v) = v_{x\overline{x}} - v_{\overline{t}}$ then*

$$L(fg) = f L(g) + g L(f) + f_x g_x + f_{\overline{x}} g_{\overline{x}} + h f_{\overline{t}} g_{\overline{t}}$$

Proof. The proof is elementary, but it is included for completeness. Derive the identities

$$(fg)_{x\overline{x}} = f g_{x\overline{x}} + g f_{x\overline{x}} + f_x g_x + f_{\overline{x}} g_{\overline{x}}$$
$$(fg)_{\overline{t}} = f g_{\overline{t}} + f_{\overline{t}} E_t^{-1} g.$$

Then

$$\begin{aligned} L(fg) &= (fg)_{x\overline{x}} - (fg)_{\overline{t}} \\ &= f(g_{x\overline{x}} - g_{\overline{t}}) + g(f_{x\overline{x}} - f_{\overline{t}}) + f_{\overline{t}}(g - E_t^{-1}g) + f_x g_x + f_{\overline{x}} g_{\overline{x}} \\ &= fL(g) + gL(f) + f_x g_x + f_{\overline{x}} g_{\overline{x}} + h f_{\overline{t}} g_{\overline{t}} \end{aligned}$$

and we are done.

7.16 Corollary. *If $L(v) = v_{x\overline{x}} - v_{\overline{t}} = 0$ then*

$$L(v^2) = v_x^2 + v_{\overline{x}}^2 + h v_{\overline{t}}^2 \geq 0$$

Proof. The corollary is an immediate consequence of the previous lemma 7.15, simply insert $f = g = v$.

7.17 Remark. Please note the difference between $(v_x)^2$ and $(v^2)_x$. Evaluating at (mh, nk) we find that

$$(v_x)^2 = \left(\frac{v_{m+1}^n - v_m^n}{h}\right)^2 = \frac{\left(v_{m+1}^n\right)^2 - 2v_{m+1}^n v_m^n + \left(v_m^n\right)^2}{h^2}$$

whereas

$$(v^2)_x = \frac{\left(v_{m+1}^n\right)^2 - \left(v_m^n\right)^2}{h}.$$

I shall write v_x^2 instead of $(v_x)^2$ for short.

7.18 Remark. The standard explicit method $\tilde{L}(v) = v_{x\overline{x}} - v_t$ obeys a similar rule. If $\tilde{L}(v) = 0$ then

$$\tilde{L}(v^2) = v_x^2 + v_{\overline{x}}^2 - h v_t^2$$

and so in general we can not be sure that $\tilde{L}(v^2) \geq 0$. This is the reason why we do not apply the explicit method to our current problem.

7.19 Lemma. *If $L(v) = v_{x\overline{x}} - v_{\overline{t}} = 0$ then $L(v_x) = 0$.*

Proof. The proof is immediate. By definition

$$L(v_x) = v_{x\overline{x}x} - v_{\overline{t}x} = (v_{x\overline{x}} - v_{\overline{t}})_x = 0$$

and we are done.

Let $L(v) = 0$ and let us begin the calculation of $L(z)$ by applying lemma 7.15 :

$$\begin{aligned} L(z) &= L(v_x^2 F + Cw) \\ &= v_x^2 L(F) + FL(v_x^2) + (v_x^2)_x F_x + (v_x^2)_{\overline{x}} F_{\overline{x}} + h(v_x^2)_{\overline{t}} F_{\overline{t}} + CL(w) \end{aligned}$$

Please take a moment to consider the structure of $L(w)$. Since

$$L(w) = L(E_x v^2) + L(E_t^{-1} v^2) + L(E_x^{-1} v^2)$$

then by corollary 7.16

$$\begin{aligned} L(E_x v^2) &= E_x v_x^2 + E_x v_{\overline{x}}^2 + h E_x v_{\overline{t}}^2 \\ L(E_t^{-1} v^2) &= E_t^{-1} v_x^2 + E_t^{-1} v_{\overline{x}}^2 + h E_t^{-1} v_{\overline{t}}^2 \\ L(E_x^{-1} v^2) &= E_x^{-1} v_x^2 + E_x^{-1} v_{\overline{x}}^2 + h E_x^{-1} v_{\overline{t}}^2 \end{aligned}$$

and it is apparent that $L(w) \geq 0$. Our aim is to bound $L(v_x^2 F)$ from below with a bounded function times a sum of appropriately shifted squares of v_x.

Returning to the expression for $L(v_x^2 F)$ we apply corollary 7.16 to the factor $L(v_x^2)$

$$L(v_x^2) = v_{xx}^2 + v_{x\overline{x}}^2 + h v_{x\overline{t}}^2.$$

The terms $(v_x^2)_x, (v_{\overline{x}}^2)_x$ and $h(v_x^2)_{\overline{t}}$ will be rewritten in the following fashion

$$\begin{aligned} (v_x^2)_x &= (E_x v_x) v_{xx} + v_x v_{xx} = (v_x + E_x v_x) v_{xx} \\ (v_x^2)_{\overline{x}} &= E_x^{-1} (v_x^2)_x = (v_x + E_x^{-1} v_x) v_{x\overline{x}} \\ h(v_x^2)_{\overline{t}} &= v_x^2 - E_t^{-1} v_x^2. \end{aligned}$$

In summary

$$\begin{aligned} L(v_x^2 F) = {} & v_x^2 L(F) + F(v_{xx}^2 + v_{x\overline{x}}^2 + h v_{x\overline{t}}^2) \\ & + F_x (v_x + E_x v_x) v_{xx} + F_{\overline{x}} (v_x + E_x^{-1} v_x) v_{x\overline{x}} + F_{\overline{t}} (v_x^2 - E_t^{-1} v_x^2). \end{aligned}$$

We split this expression into several smaller groups and focus on them independently. Consider the terms involving v_{xx} and v_x.

$$F v_{xx}^2 + F_x (v_x + E_x v_x) v_{xx}.$$

Now F_x is somewhat similar to F. By direct calculation

$$F_x = 2t(x^2 - \alpha^2)(2x + h) + t(2x + h)^2 h.$$

We focus on the term containing the common factor of $(x^2 - \alpha^2)$ and complete the square

$$\begin{aligned} & F v_{xx}^2 + 2t(x^2 - \alpha^2)(2x + h)(v_x + E_x v_x) v_{xx} \\ & \quad = t\left((x^2 - \alpha^2) v_{xx} + (2x + h)(v_x + E_x v_x)\right)^2 - t(2x + h)^2 (v_x + E_x v_x)^2 \\ & \quad \geq -2t(2x + h)^2 (v_x^2 + E_x v_x^2), \end{aligned}$$

because $(v_x + E_x v_x)^2 \leq 2v_x^2 + 2E_x v_x^2$.

Therefore

$$\begin{aligned} &Fv_{xx}^2 + F_x(v_x + E_x v_x)v_{xx} \\ &\qquad \geq -2t(2x+h)^2(v_x^2 + E_x v_x^2) + t(2x+h)^2 h(v_x + E_x v_x)v_{xx}. \end{aligned}$$

Rewriting $h(v_x + E_x v_x)v_{xx}$ as $h(v_x^2)_x = E_x v_x^2 - v_x^2$ allows us to continue with

$$\begin{aligned} &= -2t(2x+h)^2(v_x^2 + E_x v_x^2) + t(2x+h)^2(E_x v_x^2 - v_x^2) \\ &\qquad = -t(2x+h)^2\left(2v_x^2 + 2E_x v_x^2 - E_x v_x^2 + v_x^2\right) \\ &\qquad\qquad = -t(2x+h)^2\left(3v_x^2 + E_x v_x^2\right). \end{aligned}$$

Consider similarly the terms involving $v_{x\overline{x}}$ and v_x.

$$Fv_{x\overline{x}}^2 + F_{\overline{x}}(v_x + E_x^{-1}v_x)v_{x\overline{x}}$$

$F_{\overline{x}}$ is also somewhat similar to F. The shortcut is to replace h with $-h$ in the expression for F_x :

$$F_{\overline{x}} = 2t(x^2 - \alpha^2)(2x - h) - t(2x - h)^2 h.$$

Again we focus on the common factor of $(x^2 - \alpha^2)$ and complete the square

$$\begin{aligned} &Fv_{x\overline{x}}^2 + 2t(x^2 - \alpha^2)(2x - h)(v_x + E_x^{-1}v_x)v_{x\overline{x}} \\ &\qquad = t\left((x^2 - \alpha^2)v_{x\overline{x}} + (2x - h)(v_x + E_x^{-1}v_x)\right)^2 \\ &\qquad\qquad - t(2x - h)^2(v_x + E_x^{-1}v_x)^2. \end{aligned}$$

Therefore

$$\begin{aligned} &Fv_{x\overline{x}}^2 + F_{\overline{x}}(v_x + E_x^{-1}v_x)v_{x\overline{x}} \geq \\ &\qquad - t(2x - h)^2(v_x + E_x^{-1}v_x)^2 - t(2x - h)^2 h(v_x + E_x^{-1}v_x)v_{x\overline{x}} \\ &\qquad\qquad \geq -t(2x - h)^2\left(3v_x^2 + E_x^{-1}v_x^2\right). \end{aligned}$$

In summary

$$\begin{aligned} &L(z) \geq CL(w) + v_x^2 L(F) \\ &\qquad - t(2x+h)^2\left(3v_x^2 + E_x v_x^2\right) - t(2x-h)^2\left(3v_x^2 + E_x^{-1}v_x^2\right) \\ &\qquad\qquad + hFv_{x\overline{t}}^2 + F_{\overline{t}}(v_x^2 - E_t^{-1}v_x^2). \end{aligned}$$

Now F and $F_{\overline{t}}$ are both positive and we may therefore discard the terms $hFv_{x\overline{t}}^2$ and $F_{\overline{t}}v_x^2$. We need to dominate terms containing $v_x^2, E_x v_x^2, E_x^{-1}v_x^2$ and $E_t^{-1}v_x^2$, but such terms are found in abundance within the expression for $L(w)$. We discard the rest and estimate

$$L(w) \geq \left(E_x v_x^2 + v_x^2 + E_x^{-1}v_x^2 + E_t^{-1}v_x^2\right).$$

Thus

$$L(z) \geq \left(C + L(F) - 3t(2x+h)^2 - 3t(2x-h)^2\right) v_x^2 + \left(C - t(2x+h)^2\right) E_x v_x^2 + \left(C - t(2x-h)^2\right) E_x^{-1} v_x^2 + (C - F_{\overline{t}})\, E_t^{-1} v_x^2.$$

We claim that it is possible to choose $C > 0$ such that $L(z) \geq 0$ at all internal grid points (x,t) of R and sufficiently small h.

Consider the map

$$(x,t,h) \to L(F) - 3t(2x+h)^2 - 3t(2x-h)^2$$

$L(F) = F_{x\overline{x}} - F_{\overline{t}}$ is already bounded independently of (x,t,h). Recall that

$$F_{x\overline{x}}(x,t) - F_{\overline{t}}(x,t) = \frac{\partial^2 F}{\partial x^2}(\xi, t) - \frac{\partial F}{\partial t}(x, \tau)$$

for some ξ between $x-h$ and $x+h$ and τ between t and $t-h$. Now given a $h_0 > 0$ we enclose

$$R = [-\alpha, \alpha] \times [0, t_1]$$

in the larger rectangle

$$\tilde{R} = [-\alpha - h_0, \alpha + h_0] \times [-h_0, t_1].$$

Now since the continuous functions $\frac{\partial^2 F}{\partial x^2}$ and $\frac{\partial F}{\partial t}$ are bounded on the compact set $\tilde{R}$ it follows that $L(F)$ is bounded for all $(x,t,h) \in R \times (0, h_0]$.

The map

$$(x,t,h) \to -3t(2x+h)^2 - 3t(2x-h)^2$$

is merely a continuous function and will be bounded on the compact domain $R \times [0, h_0]$. Thus we discovered that

$$(x,t,h) \to L(F) - 3t(2x+h)^2 - 3t(2x-h)^2$$

is bounded on $R \times (0, h_0]$ and we may choose $C_1 > 0$ such that

$$C_1 + L(F) - 3t(2x+h)^2 - 3t(2x-h)^2 \geq 0$$

for all $(x,t,h) \in R \times (0, h_0]$.

Proceeding with this particular h_0 we turn to the maps

$$(x,t,h) \to -t(2x+h)^2$$
$$(x,t,h) \to -t(2x-h)^2.$$

They are obviously continuous and are therefore bounded on $R \times [0, h_0]$. Thus we may choose $C_2 > 0$ depending on h_0 so that

$$C_2 - t(2x+h)^2 \geq 0 \quad \text{and} \quad C_2 - t(2x-h)^2 \geq 0$$

for all $(x, t, h) \in R \times [0, h_0]$.

The final factor of $F_{\overline{t}}$ is bounded uniformly on R by the values of the corresponding smooth derivative $\frac{\partial F}{\partial t}$ on the slightly larger rectangle

$$[-\alpha, \alpha] \times [0 - h_0, t_1].$$

Thus we may choose $C_3 > 0$ such that

$$C_3 - F_{\overline{t}} \geq 0.$$

Thus we may pick $C = \max\{C_1, C_2, C_3\}$ so large that

$$L(z) \geq 0$$

and we are done. It is important to realise that C does not depend on v and $h < h_0$, but may be calculated in terms of h_0 and R.

This finally completes the proof of lemma 7.12.

As in the case of the pure initial value problem we need to bound not only v_x, but also

$$v_{xx}, v_{xxx}, v_{xxxx}, v_{\overline{t}}, v_{\overline{t}x}, v_{\overline{t}xx}, v_{\overline{tt}}.$$

Fortunately the difference equation itself offers a shortcut. Since

$$\begin{aligned} v_{\overline{t}} &= E_x^{-1} v_{xx} \\ v_{\overline{t}x} &= E_x^{-1} v_{xxx} \\ v_{\overline{t}xx} &= E_x^{-1} v_{xxxx} \end{aligned}$$

and

$$v_{\overline{tt}} = (v_{\overline{t}})_{\overline{t}} = (v_{x\overline{x}})_{\overline{t}} = (v_{\overline{t}})_{x\overline{x}} = v_{x\overline{x}x\overline{x}} = E_x^{-2} v_{xxxx}$$

it is enough to bound $v_x, v_{xx}, v_{xxx}, v_{xxxx}$.

We need the following lemma that will allow us to apply theorem 7.14 repeatedly.

7.20 Lemma. *(Slight extension lemma) If R_1 is a closed μ-rectangle such that $R_1 \cap \delta'\Omega = \emptyset$ then there exists a closed ν-rectangle R_2 with $\nu \geq \mu$ such that $R_1 \subset R_2^\circ$ and $R_2 \cap \delta'\Omega = \emptyset$.*

Proof. Write $R_1 = [x_1, x_2] \times [t_1, t_2]$. By compactness $\operatorname{dist}(R_1, \delta'\Omega) > 0$ and the sets

$$\cup_{p \in R_1} B(p, \rho/2) \quad \text{and} \quad \cup_{q \in \delta'\Omega} B(q, \rho/2)$$

are disjoint. Within $\cup_{p \in R_1} B(p, \rho/2)$ there is plenty of room to construct R_2. Pick $\nu \geq \mu$ such that $h_\nu < \frac{1}{\sqrt{2}} \frac{\rho}{2}$. Then

$$R_1 = [x_1, x_2] \times [t_1, t_2] \subset [x_1 - h_\nu, x_2 + h_\nu] \times [t_1 - h_\nu, t_2 + h_\nu] = R_2.$$

R_2 is a ν-rectangle because by lemma 7.8 R_1 is a ν-rectangle for all $\nu \geq \mu$.

Now suppose that we are given a closed μ-rectangle $R \subset \overline{\Omega} - \delta'\Omega$ along with the task of estimating $v_x, v_{xx}, v_{xxx}, v_{xxxx}$ on R independently of μ. Then we apply the slight extension lemma repeatedly and generate a series of closed ν-rectangles $(\nu \geq \mu)$: R_1, R_2, R_3, R_4 such that

$$R = R_1 \subset R_2 \subset R_3 \subset R_4$$

and $R_i \subset R^\circ_{i+1}$, $i = 1, 2, 3$ and $R_i \cap \delta'\Omega = \emptyset$, $i = 1, 2, 3, 4$. The rectangles R_2, R_3, R_4 might easily extend beyond the final line in which case we replace them by their intersection with the closed half plane

$$\{(x, t) \in \mathbb{R}^2 \; : \; t \leq T\}.$$

In any event the parabolic boundaries of the resulting ν-rectangles will be disjoint.

Then by repeated application of theorem 7.14 we gain constants C_1, C_2, C_3, C_4 that depend exclusively on the geometry, such that

$$\begin{aligned}
\|v_x\|_{R_4,\infty} &\leq C_4 \|v\|_{\overline{\Omega},\infty} \leq C_4 \|g\|_\infty \\
\|v_{xx}\|_{R_3,\infty} &\leq C_3 \|v_x\|_{R_4,\infty} \\
\|v_{xxx}\|_{R_2,\infty} &\leq C_2 \|v_{xx}\|_{R_3,\infty} \\
\|v_{xxxx}\|_{R_1,\infty} &\leq C_1 \|v_{xxx}\|_{R_2,\infty}.
\end{aligned}$$

The trick is to recall that if $L(v) = 0$, then $L(v_x) = 0$. Let C be the constant

$$C = \max\{\, C_4, C_4C_3, C_4C_3C_2, C_4C_3C_2C_1 \,\},$$

then $C\|g\|_\infty$ bounds $v_x, v_{xx}, v_{xxx}, v_{xxxx}$ independently of ν on the innermost rectangle $R = R_1$ for all sufficiently large ν.

We shall now generate a solution to the heat equation from the grid functions v^ν.

We shall use corollary 2.4. The sets $\Sigma_\nu \cap (\Omega \cup \delta''\Omega)$ constitute an increasing sequence and their union is the countable set $\Sigma \cap (\Omega \cup \delta''\Omega)$. The functions

$$v^\nu : \Sigma_\nu \cap (\Omega \cup \delta''\Omega) \to \mathbb{R}$$

are bounded at each point and our bound $\|v^\nu\| \leq \|g\|_\infty$ is even independent of ν. By corollary 2.4 we may extract a subsequence $\{v^{\nu'}\}$ such that given $p \in \Sigma \cap (\Omega \cup \delta''\Omega)$ the sequence $\{v^{\nu'(p)}\}$ will be defined (for ν' sufficiently large) and convergent. We may therefore define a function

$$u : \Sigma \cap (\Omega \cup \delta''\Omega) \to \mathbb{R}$$

through the relation

$$u(x, t) = \lim_{\nu' \to \infty} v^{\nu'}(x, t).$$

Consider now the sequence $\{v_x^\nu\}$. The grid function v_x^ν is not defined on the entire set $\Sigma_\nu \cap (\Omega \cup \delta''\Omega)$ since v^ν need not be defined at the point $(x + h_\nu, t)$. But v_x^ν is certainly defined on the set

$$A_\nu = \{(x,t) \in \Sigma_\nu \cap (\Omega \cup \delta''\Omega) \mid (x + h_\nu, t) \in \Sigma_\nu \cap (\Omega \cup \delta''\Omega)\}.$$

We claim that $A_1 \subset A_2 \subset A_3 \ldots$ and that

$$\cup_{\nu=0}^{\infty} A_\nu = \Sigma \cap (\Omega \cup \delta''\Omega).$$

Pick $(x,t) \in A_\mu$. Let $\nu \geq \mu$. Then $(x,t) \in \Sigma_\mu$ by assumption and $(x,t) \in \Sigma_\nu$ since the grids are formed by subdivision. By assumption

$$(x,t),\ (x + h_\mu, t) \in \Omega \cup \delta''\Omega.$$

so that

$$\phi_1(t) < x < x + h_\nu < x + h_\mu < \phi_2(t)$$

and $(x + h_\nu, t) \in \Omega \cup \delta''\Omega$ by the definition of Ω. Thus $(x,t) \in A_\nu$.

Now let $(x,t) \in \Sigma \cap (\Omega \cup \delta''\Omega)$. We must show that $(x,t) \in A_\nu$ for some ν. Now while $(x,t) \in \Sigma_\mu$ for some particular value of μ, the point $(x + h_\mu, t)$ need not be in $\Omega \cup \delta''\Omega$. However we can find $\nu \geq \mu$ such that $h_\nu < \phi_2(t) - x$ and then $(x + h_\nu, t) \in \Omega \cup \delta''\Omega$, so that $(x,t), (x + h_\nu, t) \in \Sigma_\nu \cap (\Omega \cup \delta''\Omega)$.

Now consider the functions

$$v_x^\nu : A_\nu \to \mathbb{R}.$$

Let $p \in \Omega \cup \delta''\Omega$. Then $p \in A_\mu$ for some value of μ and the set

$$\{v_x^\nu(p) \mid \nu \geq \mu\}$$

is bounded although our bound depends on p. But we can still apply the corollary 2.4 and extract from $\{v^{\nu'}\}$ a subsequence $\{v^{\nu''}\}$ such that for all p the sequence $\{v_x^{\nu''}(p)\}$ is defined for all sufficiently large ν'' and convergent. We may therefore define a function

$$u' : \Sigma \cap (\Omega \cup \delta''\Omega) \to \mathbb{R}$$

through the relation

$$u'(x,t) = \lim_{\nu'' \to \infty} v_x^{\nu''}(x,t).$$

We treat the sequences $\{v_{\overline{t}}^\nu\}$ and $\{v_{x\overline{x}}^\nu\}$ in a similar fashion. Thus by passing to a subsequence we may assume that given $p \in \Sigma \cap (\Omega \cup \delta''\Omega)$ the sequences $\{v_{\overline{t}}^{\nu'''}(p)\}$ and $\{v_{x\overline{x}}^{\nu'''}(p)\}$ will be defined and convergent and we may define functions

$$\dot{u}, u'' : \Sigma \cap (\Omega \cap \delta''\Omega) \to \mathbb{R}$$

by

$$\dot{u}(x,t) = \lim_{\nu''' \to \infty} v_{\overline{t}}^{\nu'''}(x,t)$$

$$u''(x,t) = \lim_{\nu''' \to \infty} v_{x\overline{x}}^{\nu'''}(x,t).$$

Now consider the functions

$$u, \dot{u}, u', u'' : \Sigma \cap (\Omega \cup \delta''\Omega) \to \mathbb{R}.$$

They are densely defined, *locally* Lipschitz continuous functions and as such they admit unique extensions to the entire set $\Omega \cup \delta''\Omega$. As an example we consider the case of u. Let $(x,t) \in \Sigma \cap (\Omega \cup \delta''\Omega)$ be given at random. Cover (x,t) with a closed μ-rectangle $R \subset \Omega \cup \delta''\Omega$. Then

$$|v_x^\nu(x,t)|,\ |v_{\bar{t}}^\nu(x,t)| \le C_R \|g\|_\infty$$

for all $(x,t) \in R \cap \Sigma_\nu$ and ν sufficiently large. As in chapter 2 we discover that the restriction

$$u : \Sigma \cap R \to \mathbb{R}$$

of u to R is Lipschitz continuous with a Lipschitz constant $L \le \sqrt{2} C_R \|g\|_\infty$. Thus u is locally Lipschitz continuous. The extension of u is easy to perform. Let $(x,t) \in R - \delta''R$ and let $(x_n, t_n) \subset \Sigma \cap (\Omega \cup \delta''\Omega)$ be any sequence of points such that

$$(x_n, t_n) \to (x,t).$$

Then there exists N such that $(x_n, t_n) \in R - \delta''R$ for $n \ge N$ and the sequence $\{u(x_n, t_n)\}_{n=N}^\infty$ is a Cauchy-sequence because

$$|u(x_n, t_n) - u(x_m, t_m)| \le L \|(x_n, t_n) - (x_m, t_m)\|.$$

We extend u by setting

$$u(x,t) = \lim_{n \to \infty} u(x_n, t_n).$$

We note to our satisfaction that the value of $u(x,t)$ is independent of our particular choice of sequence.

The function $u : \Omega \to \mathbb{R}$ is also differentiable and $\frac{\partial u}{\partial t} = \dot{u}$ while $\frac{\partial u}{\partial x} = u'$. The function $\frac{\partial u}{\partial x} : \Omega \to \mathbb{R}$ is differentiable with respect to x and $\frac{\partial^2 u}{\partial x^2} = u''$ and the derivatives extend continuously to the final line of $\delta''\Omega$. We see this by applying the methods of chapter 2 to a closed μ-rectangle around each point $p \in \Omega \cup \delta''\Omega$. Obviously u solves the heat equation within Ω and even on the final line of $\delta''\Omega$.

We claim that u is consistent with the initial-boundary conditions in the sense that

$$u(x,t) \to g(p) \quad \text{as} \quad (x,t) \to p \quad \text{with} \quad (x,t) \in \Omega \cup \delta''\Omega.$$

We begin with the easy case of

7.21 Lemma. *Let $p = (x_0, 0)$ belong to the initial line of $\delta'\Omega$. Then*

$$u(x,t) \to g(p) \quad \text{as} \quad (x,t) \to p,\ (x,t) \in \Omega$$

Proof. We shall need the auxiliary function w defined by

$$w(x,t) = (x - x_0)^2 + 3t.$$

The relevant properties of w are $w_{\overline{t}} = 3, w_{x\overline{x}} = 2$, such that

$$L(w) = w_{x\overline{x}} - w_{\overline{t}} < 0.$$

Pick $\epsilon > 0$. Then we have $\delta > 0$ such that

$$(x,t) \in B(p, 2\delta) \cap \delta'\Omega \Rightarrow |g(x,t) - g(p)| < \epsilon.$$

If $(x,t) \in \delta'\Omega_h \cap B(p,\delta)$ then $\operatorname{dist}((x,t), \delta'\Omega) < \delta$ because $\|(x,t) - p\| < \delta$ and $p \in \delta'\Omega$ and therefore

$$|g_h(x,t) - g(p)| < \epsilon.$$

Pick $C > 0$ such that $Cw(x,t) > 2\|g\|_\infty$ for all $(x,t) \in \overline{\Omega} - B(p,\delta)$. This is easy, because

$$w(x,t) = (x - x_0)^2 + 3t \geq (x - x_0)^2 \geq \delta^2.$$

Now consider the auxiliary functions ϕ, ψ defined by

$$\phi = g(p) - \epsilon - Cw - v^\nu \quad \text{and} \quad \psi = v^\nu - g(p) - \epsilon - Cw.$$

We claim that these functions are non-positive at all grid points in $\overline{\Omega}_\nu$. Let us first investigate the consequence of our claim :

$$g(p) - \epsilon - Cw(x,t) \leq v^\nu(x,t) \leq Cw(x,t) + g(p) + \epsilon$$

at all grid points (x,t). Letting ν tend to infinity gives

$$g(p) - \epsilon - Cw(x,t) \leq u(x,t) \leq g(p) + Cw(x,t) + \epsilon$$

at all grid points (x,t). By the continuity of u and the denseness of Σ this inequality is immediately extended to the entire set Ω. Letting (x,t) tend to p from within Ω gives

$$g(p) - \epsilon \leq \liminf u(x,t) \leq \limsup u(x,t) \leq g(p) + \epsilon.$$

Since ϵ was picked at random this will complete the proof. We shall now prove our claims regarding ϕ and ψ. Please note that $L(\phi) = -CL(w) > 0$. By corollary 7.4 it is enough to show that $\phi(x,t) \leq 0$ at all grid points on the parabolic boundary. Let therefore $(x,t) \in \delta'\Omega_h$. There are two cases to consider: either $(x,t) \in B(p,\delta)$ or $(x,t) \notin B(p,\delta)$.

If $(x,t) \in B(p,\delta)$ then

$$\phi(x,t) = g(p) - \epsilon - Cw(x,t) - v^\nu(x,t) \leq |g(p) - v^\nu(x,t)| - \epsilon$$

because $w \geq 0$ and trivially $g(p) - v^\nu(x,t) \leq |g(p) - v^\nu(x,t)|$. But since $(x,t) \in \delta'\Omega_h$, $v^\nu(x,t) = g_h(x,t)$ and

$$|g(p) - v^\nu(x,t)| = |g(p) - g_h(x,t)| < \epsilon$$

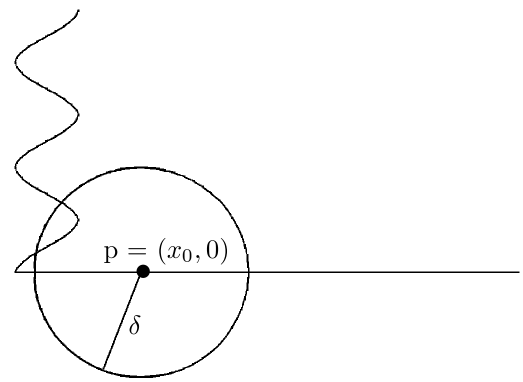

Figure 7.1: Geometry for the proof of lemma 7.21.

so that $\phi(x,t) < 0$ and we are done.

If $(x,t) \notin B(p,\delta)$ then by virtue of the choice of C we have

$$\phi \leq \|g\|_\infty - \epsilon - 2\|g\|_\infty + \|g\|_\infty = -\epsilon < 0.$$

The function ψ is handled in a similar manner and we are done.

We now consider a point $p = (x_1, t_1)$ on the parabolic boundary with $t_1 > 0$. We need the following definition

7.22 Definition. *(Barrier function) A barrier function for the point p is a function $v_p : U_p \to \mathbb{R}$ defined on the intersection between $\overline{\Omega}$, the closed half plane of $t \leq t_1$ and some open ball $B(p,\rho)$,*

$$U_p = \overline{\Omega} \cap \{(x,t) \in \mathbb{R}^2 \,:\, t \leq t_1\} \cap B(p,\rho)$$

such that

- $v_p \in C(U_p)$
- $v(P) \geq 0$ *for all* $P \in U_p$ *and* $v(P) = 0$ *iff* $P = p$.
- $v_{xx} - v_{\overline{t}} < 0$ *at all grid points within* U_p *and* h *sufficiently small.*

7.23 Remark. Barrier functions can be found if ϕ_1, ϕ_2 are Lipschitz-continuous, but even less is required. Barrier functions for the heat equation are given in Petrowski's book [2] and we shall not dwell on the question of generating specific examples. The curious restriction of $t \leq t_1$ is deliberate ! Petrowski's barrier-functions can not be used for $t > t_1$. I tried and failed to generate specific functions that were up to this extended task. Instead we follow Petrowski's analysis and split the treatment of $(x,t) \to p$ into two cases namely

- $(x, t) \to p$ with $t \leq t_1$ and
- $(x, t) \to p$ with $t \geq t_1$

We have the following lemma

7.24 Lemma. *If $p = (x_1, t_1) \in \delta'\Omega$ with $t_1 > 0$ has a barrier function then*

$$u(x,t) \to g(p) \quad \textit{as} \quad (x,t) \to p \quad \textit{with} \quad (x,t) \in \Omega \cup \delta''\Omega \quad \textit{and} \quad t \leq t_1$$

Proof. The proof of this lemma is quite similar to the proof of the previous lemma. We may assume that our point p lies on the left hand side of the parabolic boundary, so let $p = (x_1, t_1) = (\phi_1(t_1), t_1)$ with $t_1 > 0$. Let

$$v_p : U_p \to \mathbb{R}$$

be a barrier function for p. Let $\epsilon > 0$ be given. Then there exists $\delta > 0$ such that

$$(x,t) \in B(p, 2\delta) \cap \delta'\Omega \Rightarrow |g(x,t) - g(p)| < \epsilon.$$

Then

$$(x,t) \in B(p, \delta) \cap \delta'\Omega_\nu \Rightarrow |g_\nu(x,t) - g(p)| < \epsilon.$$

Set

$$U_\epsilon = B(p, \delta) \cap \overline{\Omega}.$$

Now choose $\alpha > 0$ so small that the *compact* region D_α bounded by the lines $t = t_1$, $t = t_1 - \alpha$, $x = x_1 + \alpha$ and $x = \phi_1(t)$, $t_1 - \alpha \leq t \leq t_1$ is contained in U_p. We may assume that U_ϵ is so small that it does not completely contain D_α. Now pick $C_1 > 0$ such that $C_1 v_p(x,t) \geq 2\|g\|_\infty$ for all $(x,t) \in D_\alpha - U_\epsilon$. This is certainly possible because $D_\alpha - U_\epsilon$ is compact and v_p is continuous and positive apart from the single zero in p which is well contained within U_ϵ. As in the previous proof we consider the auxiliary functions of

$$\phi = g(p) - \epsilon - Cv_p - v^\nu \quad \text{and} \quad \psi = v^\nu - g(p) - \epsilon - Cv_p.$$

We claim that these functions are non-positive at all grid points in D_α. Again it will suffice to consider the parabolic boundary points of D_α. Let (x,t) be such a point. Then at least one of the adjacent points $(x - h_\nu, t), (x, t - h_\nu), (x + h_\nu, t)$ is not contained within D_α. If $(x - h_\nu, t) \notin D_\alpha$ then $(x,t) \in \delta'\Omega_\nu$ and $v^\nu(x,t) = g_\nu(x,t)$. There are two possibilities either $(x,t) \in U_\epsilon$ or $(x,t) \notin U_\epsilon$. If $(x,t) \in U_\epsilon$ then

$$\begin{aligned}\phi(x,t) &= g(p) - \epsilon - Cv_p(x,t) - v^\nu(x,t) \\ &\leq |g(p) - v^\nu(x,t)| - \epsilon = |g(p) - g_\nu(x,t)| - \epsilon < 0\end{aligned}$$

because $v_p \geq 0$ and $g(p) - v^\nu(x,t) \leq |g(p) - v^\nu(x,t)|$. On the other hand if $(x,t) \notin U_\epsilon$ then

$$\begin{aligned}\phi(x,t) &= g(p) - \epsilon - C_1 v_p(x,t) - v^\nu(x,t) \\ &\leq \|g\|_\infty - \epsilon - 2\|g\|_\infty + \|g\|_\infty = -\epsilon < 0\end{aligned}$$

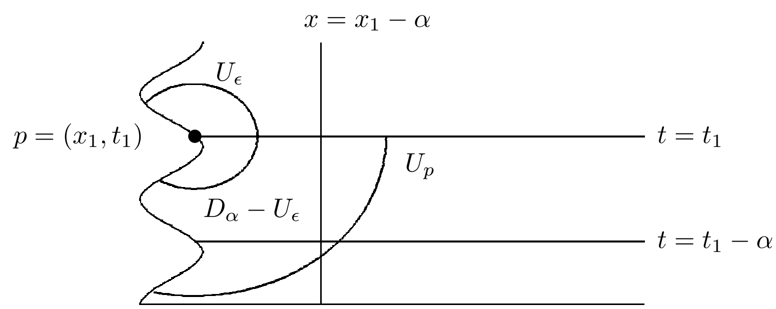

Figure 7.2: Geometry for the proof of lemma 7.24.

by virtue of the choice of C_1. Finally if $(x, t - h_\nu)$ or $(x, t + h_\nu) \notin D_\alpha$ then $(x, t) \notin U_\epsilon$ and we proceed as before.

In summary: ϕ is non-positive. ψ is treated in a similar manner and we have the inequality

$$g(p) - \epsilon - Cv_p(x, t) \leq v^\nu(x, t) \leq Cv_p(x, t) + g(p) + \epsilon$$

at all grid points $(x, t) \in D_\alpha$.

The rest of the proof is completed in the same manner as the proof of the previous lemma and we find that

$$u(x, t) \to g(p) \quad \text{as} \quad (x, t) \to p \quad \text{with} \quad (x, t) \in \Omega \cup \delta''\Omega \quad \text{and} \quad t \leq t_1.$$

7.25 Remark. Please note for future use that this implies the existence of a neighbourhood $U_{2\epsilon} = \overline{\Omega} \cap B(p, r_\epsilon)$ such that

$$g(p) - 2\epsilon \leq v^\nu(x, t) \leq g(p) + 2\epsilon$$

for all sufficiently large ν and for all grid points

$$(x, t) \in U_{2\epsilon} \cap \{\, (x, t) \in \mathbb{R}^2 \,:\, t \leq t_1 \,\}.$$

It is merely a matter of utilising the fact that $w(x, t) \to 0$ as $(x, t) \to p$ with $t \leq t_1$. We may assume that $U_{2\epsilon} \subset U_\epsilon$.

We now turn to the final case of

7.26 Lemma. *If $p = (x_2, t_2) \in \delta'\Omega$ with $t_2 > 0$ then*

$$u(x, t) \to g(p) \quad \textit{as} \quad (x, t) \to p \quad \textit{with} \quad (x, t) \in \Omega \quad \textit{and} \quad t \geq t_2.$$

Proof. We may assume that our point p lies on the left hand side of the parabolic boundary, i.e. $p = (x_2, t_2) = (\phi_1(t_2), t_2)$. We need substantial preparation for this proof. Pick $\epsilon > 0$ at random. Pick $0 < \delta < t_2$. Consider $w(x,t) = (x - x_2)^2 + 3(t - t_2)$. It is possible to pick $C_2 > 0$ such that $C_2 w(x,t) \geq 2\|g\|_\infty$ for all $(x,t) \notin U_{2\epsilon}$ with $t_2 - \delta \leq t$, provided that we choose δ carefully. There are two cases to consider: either $t_2 - \delta \leq t < t_2$ or $t_2 \leq t$. If $t_2 \leq t$ then obviously

$$w(x,t) = (x - x_2)^2 + 3(t - t_2) \geq (x - x_2)^2 \geq r_\epsilon^2 > \frac{1}{2} r_\epsilon^2.$$

If $t_2 - \delta \leq t < t_2$ then

$$w(x,t) = (x - x_2)^2 + 3(t - t_2) \geq (x - x_2)^2 - 3\delta \geq r_\epsilon^2 - 3\delta > \frac{1}{2} r_\epsilon^2,$$

provided that we choose $\delta < \frac{1}{6} r_\epsilon^2$. Thus in either case $w(x,t) > \frac{1}{2} r_\epsilon^2$ if $(x,t) \notin U_{2\epsilon}$ with $t \geq t_2 - \delta$. Thus

$$C_2 w(x,t) > \frac{1}{2} C_2 r_\epsilon^2 > 2\|g\|_\infty \quad \text{for} \quad (x,t) \notin U_{2\epsilon} \quad \text{with} \quad t_2 - \delta \leq t$$

provided that we pick $C_2 > 4\|g\|_\infty r_\epsilon^{-2}$. Obviously any smaller δ will do the trick and we pick δ such that

$$C_2 w(x,t) \geq 3C_2(t - t_2) \geq -3C_2\delta > -\epsilon$$

for *any* (x,t) with $t_2 - \delta \leq t$.

Now consider the set

$$\begin{aligned} \Omega(\delta) &= \{(x,t) \in \Omega \mid t_2 - \delta < t\} \\ &= \{(x,t) \in \mathbb{R}^2 \mid t_2 - \delta < t < T, \phi_1(t) < x < \phi_2(t)\} \end{aligned}$$

and pick ν so large that there is at least one layer of nodes between the lines $t = t_2 - \delta$ and $t = t_2$, i.e. pick $h_\nu < \delta$. As in the previous proofs we consider an auxiliary function ϕ which we claim to be negative at all grid points within $\overline{\Omega}(\delta)$. In this case ϕ is given by

$$\phi = g(p) - 3\epsilon - C_2 w - v^\nu.$$

We claim that $\phi(x,t) \leq 0$ at all the parabolic boundary nodes of $\overline{\Omega}(\delta)$ and since $L(\phi) = -C_2 L(w) > 0$ we shall be able to deduce that $\phi \leq 0$ at all nodes of $\overline{\Omega}(\delta)$.

Now let $(x,t) \in \overline{\Omega}(\delta) \cap \Sigma_\nu$ be a parabolic boundary node. Then either $(x,t) \in \delta' \Omega_h$ or (x,t) is one of the nodes in the lowermost layer of $\overline{\Omega}(\delta) \cap \Sigma_\nu$, so that $t_2 - \delta \leq t \leq t_2$.

If $(x,t) \notin U_{2\epsilon}$ then

$$\phi \leq \|g\|_\infty - 3\epsilon - 2\|g\|_\infty + \|g\|_\infty = -3\epsilon < 0$$

by virtue of the choice of C_2.

If $(x,t) \in U_{2\epsilon}$ then we estimate ϕ by

$$\phi \leq |g(p) - v^\nu| - 3\epsilon - C_2 w$$

and there are two cases to consider namely $w \geq 0$ and $w < 0$.

If $w \geq 0$ then we discard w completely and estimate

$$\phi \leq |g(p) - v^\nu| - 3\epsilon$$

and there are two sub-cases to consider namely $(x,t) \in \delta'\Omega_h$ or $(x,t) \notin \delta'\Omega_h$. If $(x,t) \in \delta'\Omega_h$ then $v^\nu = g_\nu$ and

$$\phi \leq |g(p) - v^\nu| - 3\epsilon = |g(p) - g_h| - 3\epsilon < -2\epsilon < 0$$

because we are well within U_ϵ. If $(x,t) \notin \delta'\Omega_h$ then $t \leq t_2$ and we estimate

$$\phi \leq |g(p) - v^\nu| - 3\epsilon < 2\epsilon - 3\epsilon = -\epsilon$$

because we are within $U_{2\epsilon} \cap \{\, (t,x) \in \mathbb{R}^2 \;:\; t \leq t_2 \,\}$.

If on the other hand $w < 0$ then we use that $-C_2 w \leq \epsilon$ and estimate

$$\phi \leq |g(p) - v^\nu| - 2\epsilon.$$

We have the same two sub-cases to consider: either $(x,t) \in \delta'\Omega_h$ or $(x,t) \notin \delta'\Omega_h$, but they are treated as before. Please note that in the last case we actually use up all our multiples of ϵ.

We treat the function $\psi = v^\nu - 3\epsilon - C_2 w - g(p)$ in a similar fashion and discover that $\psi \leq 0$ at all boundary nodes within $\overline{\Omega}(\delta)$. As in the previous lemmata we are left with an inequality

$$g(p) - 3\epsilon - C_2 w \leq v^\nu \leq g(p) + 3\epsilon + C_2 w$$

at all grid points within $\overline{\Omega}(\delta)$. Letting ν tend to infinity yields

$$g(p) - 3\epsilon - C_2 w \leq u \leq g(p) + 3\epsilon + C_2 w$$

at all points in $\Omega \cap \Sigma_h$. The continuity of u extends this inequality to the entire set of Ω and letting $(x,t) \to p$ eliminates w leaving us with

$$g(p) - 3\epsilon \leq \liminf u(x,t) \leq \limsup u(x,t) \leq g(p) - 3\epsilon$$

and we are finally done with this proof.

7.27 Remark. I would like the reader to reflect briefly on the similarities of the proofs of the previous three lemmata. There are some technical differences, but the main idea of exploiting the maximum principle of lemma 7.3 is the same. In each case we had an auxiliary function ϕ for which we needed to show that $\phi \leq 0$. By construction $L(\phi) > 0$ and by lemma 7.3 it was enough to consider certain parabolic boundary nodes. In chapter 8 we deal with a more general operator $\hat{L}$ for which $\hat{L}(\phi) > 0$ implies that any maxima achieved at an *internal* node will be negative ! Thus it still suffices to show explicitly that $\phi \leq 0$ at parabolic boundary nodes.

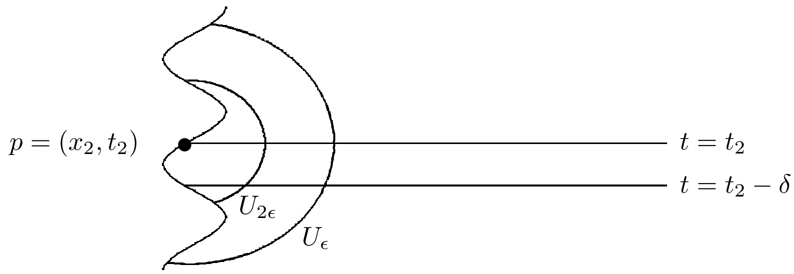

Figure 7.3: Geometry for the proof of lemma 7.26.

In summary: We have generated a function $u : \Omega \cup \delta''\Omega \to \mathbb{R}$ such that u actually satisfies the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, and the functions $u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}$ are locally Lipschitz continuous on $\Omega \cup \delta''\Omega$. Please note that u is in fact C^∞. We have already demonstrated how to bound some of the divided differences of v^ν. The rest are handled in the same fashion.

Further u satisfies the initial-boundary condition in the sense that

$$u(x,t) \to g(p) \quad \text{as} \quad (x,t) \to p \quad \text{with} \quad (x,t) \in \Omega \cup \delta''\Omega$$

for all p on the parabolic boundary $\delta'\Omega$ of Ω. Thus

$$u \in C^\infty(\Omega \cup \delta''\Omega) \cap C(\overline{\Omega})$$

and by the uniqueness theorem 5.7 this is the only possibility.

The uniqueness theorem settles a couple of questions regarding our construction of u. We extracted a convergent subsequence from bounded sequences, but were there more than one cluster point in each case, i.e. was it really necessary to pass to a proper subsequence ? The answer is no. Suppose we at some point encounter a subsequence with more than one cluster point. Then we can generate two solutions which are essential different but still satisfy the heat equation as well as the initial-boundary values. Clearly this can not be so.

The construction of the discrete boundary condition g_h contained some measure of randomness, but as long as

$$\forall\, h > 0\, \forall\, \epsilon > 0\, \exists\, \delta > 0\ :\ (x,t) \in B(A,\delta) \cap \delta'\Omega_h \Rightarrow |g(A) - g_h(x,t)| < \epsilon$$

it does not matter how we assign values to g_h.

This completes our treatment of the mixed problem for the heat equation.

Chapter 8

Mixed Problems for the General Equation

In this chapter we study the initial-boundary value problems for the differential equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + cu + f.$$

We consider sets Ω of the type

$$\Omega = \{(x,t) \in \mathbb{R}^2 \,:\, 0 < t < T,\ \phi_1(t) < x < \phi_2(t)\},$$

where $\phi_1, \phi_2 : [0,T] \to \mathbb{R}$ are continuous functions with

$$\phi_1(t) < \phi_2(t)$$

for $t \in [0,T]$ and given an initial-boundary condition $g : \delta'\Omega \to \mathbb{R}$ we seek a function $u : \Omega \cup \delta''\Omega \to \mathbb{R}$ such that the differential equation is satisfied in every point of Ω and

$$u(x,t) \to g(p) \quad \text{as} \quad (x,t) \to p \quad \text{with} \quad (x,t) \in \Omega \cup \delta''\Omega$$

for every point $p \in \delta'\Omega$.

8.1 Notation. We continue to reserve v_x for the divided difference $\delta_x v$ and write $\frac{\partial u}{\partial x}$ for the partial derivative of u with respect to x.

We make substantial assumptions on the coefficients a, b, c and the function f. We assume that they are bounded and infinitely differentiable with bounded derivatives on the strip

$$S_T = \{(x,t) \in \mathbb{R}^2 \,:\, 0 \le t \le T\}.$$

We assume that $\inf\limits_{S_T} a > 0$.

We do not consider the question of extending functions defined on Ω to S_T in the desired fashion. This is a non-trivial problem which I am quite happy to leave to a topologist. Among other things this assumption allows us to eliminate the inhomogeneous term of f by considering the pure initial value problem

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + b\frac{\partial w}{\partial x} + cw + f, \quad (x,t) \in S_T \tag{8.1}$$

with $w = 0$ on the initial line of $t = 0$ along with the modified initial value problem of

$$\frac{\partial \omega}{\partial t} = a\frac{\partial^2 \omega}{\partial x^2} + b\frac{\partial \omega}{\partial x} + c\omega, \quad (x,t) \in \Omega \cup \delta''\Omega$$

and $\omega = g - w$ on the parabolic boundary $\delta'\Omega$ of Ω.

In chapter 3 we rediscovered that equation (8.1) does in fact have a unique solution $w : S_T \to \mathbb{R}$ which is bounded and infinitely differentiable with bounded derivatives on S_T.

Thus we may assume that $f = 0$ and restrict ourselves to the case of

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + cu, \quad (x,t) \in \Omega \cup \delta''\Omega \tag{8.2}$$

with $u = g$ on $\delta'\Omega$.

In chapter 5 we made substantial use of the transformation

$$v(x,t) = u(x,t)e^{-Ct}, \quad C \in \mathbb{R}$$

which allowed us to adjust the value of the creation term c. Recall that if u satisfies equation (8.2) then

$$\begin{aligned}\frac{\partial v}{\partial t} = \left(\frac{\partial u}{\partial t} - Cu\right)e^{-Ct} = \left(a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + cu - Cu\right)e^{-Ct} \\ = a\frac{\partial^2 v}{\partial x^2} + b\frac{\partial v}{\partial x} + (c - C)\,v\end{aligned}$$

and $v(x,t) = g(x,t)e^{-Ct}$ for $(x,t) \in \delta'\Omega$.

Thus by choosing C large enough we may assume that the creation term c is either non-positive or even very large but negative. It is important to realise that neither a nor b are affected by the transformation and that u can be recovered from v.

We proceed essentially as in the case of the heat equation. The main problem is to bound the divided differences of our grid-functions independently of the grid.

We begin by setting up the grids and the difference equations. Pick $N \in \{1, 2, 3, \dots\}$ at random and set $k = T/N$. Pick h at random. Let $\Sigma_{h,k}$ be the grid

$$\Sigma_{h,k} = \{(jh, nk) \mid j, n \in \mathbb{Z}\}.$$

Define $\overline{\Omega}_{h,k} = \overline{\Omega} \cap \Sigma_{h,k}$. We subdivide $\overline{\Omega}_{h,k}$ into two disjoint sets $\Omega_{h,k}$ and $\delta'\Omega_{h,k}$ which we define in the following fashion. Let $(x,t) \in \overline{\Omega}_{h,k}$. **If** the adjacent points $(x-h,t), (x,t-k), (x+h,t) \in \overline{\Omega}$ **then** $(x,t) \in \Omega_{h,k}$ **else** $(x,t) \in \delta'\Omega_{h,k}$.

It is convenient to think of the $\Omega_{h.k}$ as the set of internal nodes and $\delta'\Omega_{h,k}$ as the set of parabolic boundary nodes.

We define $g_{h,k} : \delta'\Omega_{h,k} \to \mathbb{R}$ as we did in the case of the heat equation. Let $(x,t) \in \delta'\Omega_{h,k}$. Then by compactness there exists a point $(x',t') \in \delta'\Omega$ that minimizes the distance between (x,t) and $\delta'\Omega$. Set $g_{h,k}(x,t) = g(x',t')$. There might easily be more than one such point (x',t') in which case we pick one at random.

On $\overline{\Omega}_{h,k}$ we consider the following difference equation :

$$L(v) = av_{x\overline{x}} + \frac{1}{2}b(v_x + v_{\overline{x}}) + cv - v_{\overline{t}} = 0$$

at all internal nodes $\Omega_{h,k}$ and $v = g_{h,k}$ on the parabolic boundary nodes $\delta'\Omega_{h,k}$.

8.2 Remark. Unlike chapter 7 we do not limit ourselves to $h = k$. We shall eventually choose $k = \lambda h^2$, but for now we develop the theory for general h and k. We shall however insist that $k = T/N$ for some positive integer N, so that $\delta''\Omega$ lies on a grid line.

We need the following lemma on the behaviour of L.

8.3 Lemma. *There exists a $h_0 > 0$ such that $L(v) = 0$ and $c \leq 0$ implies that*

$$\max_{\overline{\Omega}_{h,k}} |v| = \max_{\delta'\Omega_{h,k}} |v|$$

for all $h \leq h_0$. Specifically we may choose $h_0 = \frac{2 \inf a}{\|b\|_\infty}$.

Proof. The difference equation $L(v) = 0$ is equivalent to

$$(1 + 2\lambda a - ck)v_j^n = \lambda \left(a + \frac{1}{2}bh\right) v_{j+1}^n + \lambda \left(a - \frac{1}{2}bh\right) v_{j-1}^n + v_j^{n-1}.$$

The proof will be by induction on the number of time-steps, so let

$$\|v^n\| = \max\{|v_j^n| \mid j \in \mathbb{Z},\ (jh, nk) \in \overline{\Omega}_{h,k}\}$$

be the maximum norm of v at the n'th time step. Obviously

$$\|v^0\| \leq \max_{\delta'\Omega_{h,k}} |v|,$$

so we assume that

$$\|v^{n-1}\| \leq \max_{\delta'\Omega_{h,k}} |v|$$

for some particular value of n and consider $\|v^n\|$. Let (jh, nk) be a grid point such that $|v_j^n| = \|v^n\|$. There are two possibilities: either $(jh, nk) \in \Omega_{h,k}$ or $(jh, nk) \in \delta'\Omega_{h,k}$. If $(jh, nk) \in \delta'\Omega_{h,k}$ then certainly

$$\|v^n\| \leq \max_{\delta'\Omega_{h,k}} |v|.$$

If $(jh, nk) \in \Omega_{h,k}$ then we apply the triangle inequality to the difference equation getting

$$\begin{aligned}&\left(1 + 2\lambda a_j^n - c_j^n k\right) \|v^n\| \leq \\ &\qquad \lambda \left(a_j^n + \frac{1}{2} b_j^n h\right) \|v^n\| + \lambda \left(a_j^n - \frac{1}{2} b_j^n h\right) \|v^n\| + \|v^{n-1}\| \\ &\qquad = 2\lambda a_j^n \|v^n\| + \|v^{n-1}\|\end{aligned}$$

provided that h is chosen so small that the coefficients

$$a_j^n \pm \frac{1}{2} b_j^n h$$

are positive for all possible values of j, n. This is easy, because

$$a_j^n \pm \frac{1}{2} b_j^n h \geq \inf a - \frac{1}{2} \|b\|_\infty h \geq 0$$

if $h \leq h_0 = \frac{2 \inf a}{\|b\|_\infty}$. The term $(1 + 2\lambda a_j^n - c_j^n k)$ is always positive because $c \leq 0$. Thus we may deduce that

$$\left(1 + 2\lambda a_j^n\right) \|v^n\| \leq \left(1 + 2\lambda a_j^n - c_j^n k\right) \|v^n\| \leq 2\lambda a_j^n \|v^n\| + \|v^{n-1}\|$$

which implies $\|v^n\| \leq \|v^{n-1}\|$ and we are done with the proof of this lemma.

As an immediate corollary we note that if $v \equiv 0$ on the parabolic boundary nodes then $v \equiv 0$ is the only solution to the difference equation. As in the case of the heat equation the difference equation is merely a finite system of linear equations. The unknowns are the values of v at each of the internal grid points and there is an equal number of equations. Thus for each choice of $g_{h,k}$ our difference equations admits precisely one solution v and we have the basic estimate

$$\max_{\overline{\Omega}_{h,k}} |v| = \max_{\delta'\Omega_{h,k}} |v| \leq \|g\|_\infty$$

which is *independent* of the actual grid.

8.4 Remark. The value $\frac{2 \inf a}{\|b\|_\infty}$ will surface so often that it is practical to define h_{max} by

$$h_{max} = \frac{2 \inf a}{\|b\|_\infty}.$$

We shall normally limit ourselves to $h \leq h_{max}$ or $h < h_{max}$. In the rare event of $\|b\|_\infty = 0$ no restrictions will be placed on h, but in general $h_{max} < \infty$ and we must tread carefully.

We now turn to the task of estimating v_x. As in chapter 7 we can not provide a global estimate of v_x but we must attempt to estimate v_x relative to v on every compact subset of $\Omega \cup \delta''\Omega$. Fortunately it still suffices to consider closed rectangles instead of general compact sets.

In order to be precise we need to generalise the concept of a μ-rectangle. Consider a sequence of grids

$$\Sigma_\nu = \{\, (jh_\nu, nk_\nu) \;:\; j \in \mathbb{Z}, n \in \mathbb{Z} \,\}, \quad \nu = 1, 2, 3, \ldots$$

that are formed by *subdivision*, i.e.

$$h_{\nu+1} = \frac{1}{M'} h_\nu$$
$$k_{\nu+1} = \frac{1}{N'} k_\nu$$

for some positive integers M' and N'.

In chapter 7 we dealt with the very special case of $M' = N' = 2$. In this chapter we shall need the case of

$$h_{\nu+1} = \frac{1}{2} h_\nu$$
$$k_{\nu+1} = \frac{1}{4} k_\nu,$$

such that $\lambda_\nu = k_\nu h_\nu^{-2}$ is a constant independent of ν, but for now we continue to develop the theory for general M' and N'.

8.5 Definition. *(μ-rectangle) A rectangle R is called a μ-rectangle if the four corners are members of Σ_μ.*

As in chapter 7 the following lemma is an immediate corollary.

8.6 Lemma. *If R is a μ-rectangle then R is a ν-rectangle for $\nu \geq \mu$.*

The following covering lemma is proven in a manner quite similar to the special case of $M = N = 2$ and $h_0 = k_0$.

8.7 Lemma. *Let $K \subset \mathbb{R}^2$ be a compact set such that $K \cap \delta'\Omega = \emptyset$. Then there exists a μ and a finite number n of closed μ-rectangles R_i such that*

$$K \subset \cup_{i=1}^{n} R_i^\circ$$

and $R_i \cap \delta'\Omega = \emptyset$. The set R_i° is the interior of R_i.

If K is a compact subset of $\Omega \cup \delta''\Omega$ we cover K by a finite number n of closed μ-rectangles $R_i, i = 1, 2, \ldots n$, such that $R_i \cap \delta'\Omega = \emptyset$. If R_i extends beyond the final line of $t = T$ we replace R_i with

$$R_i \cap \{\, (x,t) \in \mathbb{R}^2 \;:\; t \leq T \,\}.$$

By construction $t = T$ is a grid line for all ν and the truncated R_i are μ-rectangles.

As in the case of the heat equation we have the following theorem

8.8 Theorem. *Let* $\sup c < 0$. *Let* $R_1 \subset R_2 \subset R_3$ *be closed* μ*-rectangles, such that*

$$\delta' R_i \cap \delta' R_{i+1} = \emptyset \quad \textit{for} \quad i = 1, 2.$$

Then there exists $C > 0$ *such that for all* $\nu \geq \mu$ *we have the implication*

$$L(v^\nu) \equiv 0 \textit{ on } \Sigma_\nu \cap (R_2^\circ \cup \delta'' R_2) \Rightarrow \forall\, (x,t) \in R_1 \cap \Sigma_\nu \ : \ |v_x^\nu(x,t)| \leq C \max_{\Sigma_\nu \cap R_3} |v^\nu|.$$

The proof of this theorem consists mainly of an analysis of the auxiliary functions $z^\nu : \Sigma_\nu \cap R \to \mathbb{R}$ given by

$$z^\nu = (v_x^\nu)^2 F + C w^\nu$$

where $F(x,t) = t(x^2 - \alpha^2)^2$ is the familiar cut-off function for the rectangle

$$R = [-\alpha, \alpha] \times [0, T]$$

and the function w^ν is a finite sum of appropriately shifted squares of v^ν, specifically

$$w^\nu(x,t) = v^\nu(x - h_\nu, t)^2 + v^\nu(x, t - k_\nu)^2 + v^\nu(x + h_\nu, t)^2.$$

We have the following key theorem:

8.9 Theorem. *If* $\sup c < 0$ *then*

$$\exists\, C_0 > 0 \ : \forall \nu \geq \mu \ : \ L(v_\nu) = 0 \textit{ on } \Sigma_\nu \cap (R^\circ \cup \delta'' R) \Rightarrow \forall C \geq C_0 \ : \ L(z) \geq 0 \textit{ on } \Sigma_\nu \cap (R^\circ \cup \delta'' R).$$

In order to exploit this theorem we need the following lemma which is the discrete analogue of theorem 6.2.

8.10 Lemma. *If* $h \leq h_{max}$ *and* $\sup c < 0$ *then* $L(v) \geq 0$ *implies that all internal global maxima will be non-positive. In particular if* v *assumes even a single positive value then no such point can be found and the maximum is achieved only at a parabolic boundary node.*

Proof. The inequality $L(v) \geq 0$ is equivalent to

$$(1 + 2\lambda a - ck) v_j^n \leq \lambda \left(a + \frac{1}{2} bh \right) v_{j+1}^n + \lambda \left(a - \frac{1}{2} bh \right) v_{j-1}^n + v_j^{n-1}.$$

Assume that $M = \max_{\overline{\Omega}_{h,k}} v$ is achieved at an *internal* node (jh, nk), i.e. $M = v_j^n$. Then $v_{j+1}^n, v_{j-1}^n, v_j^{n-1} \leq M$ and

$$(1 + 2\lambda a - ck)M \leq \lambda\left(a + \frac{1}{2}bh\right)M + \lambda\left(a - \frac{1}{2}bh\right)M + M = (1 + 2\lambda a)M$$

provided that $h \leq h_{max}$, so that the coefficients $a_j^n \pm \frac{1}{2}b_j^n h$ are non-negative for all values of j, n. Since $c < 0$ we are left to conclude that $M \leq 0$.

The rest of the proof is trivial. If there is even a single positive value of v_j^n we can not have a global non-positive maximum anywhere.

Let us now prove that lemma 8.10 together with theorem 8.9 implies theorem 8.8. We are given three closed μ-rectangles $R_1 \subset R_2 \subset R_3$ such that

$$\delta' R_i \cap \delta' R_{i+1} = \emptyset.$$

We may assume without loss of generality that R_2 is the familiar rectangle

$$R = [-\alpha, \alpha] \times [0, T].$$

The condition $\delta' R_2 \cap \delta' R_3 = \emptyset$ places R_2 well within R_3 and ensures that

$$w^\nu(x,t) = v^\nu(x - h_\nu, t)^2 + v^\nu(x, t - k_\nu)^2 + v^\nu(x + h_\nu, t)^2$$

is bounded by

$$3\left(\max_{\Sigma_\nu \cap R_3} |v^\nu|\right)^2$$

for all $(x,t) \in \Sigma_\nu \cap R_2$.

Now by assumption

$$L(v^\nu) \equiv 0 \text{ on } \Sigma_\nu \cap (R_2^\circ \cup \delta'' R_2)$$

so by theorem 8.9 there exists a C_0 such that $L(z^\nu) \geq 0$ on $\Sigma_\nu \cap (R_2^\circ \cup \delta'' R_2)$. The auxiliary functions z^ν are clearly non-negative, so lemma 8.10 implies that the maximum of z^ν is certainly achieved on the parabolic boundary of R_2 where the cut-off function vanishes by design. If $z^\nu \not\equiv 0$ then the maximum is achieved exclusively on the parabolic boundary of R_2. In any case

$$(v_x^\nu)^2 F \leq z = (v_x^\nu)^2 F + C_0 w \leq \max_{\Sigma_\nu \cap R_2} z = C_0 \max_{\Sigma_\nu \cap \delta' R_2} w \leq 3\, C_0 \left(\max_{\Sigma_\nu \cap R_3} |v^\nu|\right)^2.$$

The condition $\delta' R_1 \cap \delta' R_2 = \emptyset$ places R_1 well within R_2 in the sense that

$$\inf_{R_1} F > 0.$$

Thus

$$|v_x^\nu(x,t)| \leq \sqrt{3\left(\inf_{R_1} F\right)^{-1} C_0 \max_{\Sigma_\nu \cap R_3} |v^\nu|}$$

for all $(x,t) \in \Sigma_\nu \cap R_1$. This completes the demonstration of the fact that theorem 8.9 and lemma 8.10 implies theorem 8.8.

Before proving theorem 8.9 we derive the following corollary of theorem 8.8 which eliminates the need for an “intermediate” rectangle R_2.

8.11 Corollary. *Let $R_1 \subset R_3$ be closed μ-rectangles, so that $\delta' R_1 \cap \delta' R_3 = \emptyset$. Then there exist a constant $C > 0$ such that for all ν strictly greater than μ we have the implication*

$$L(v^\nu) \equiv 0 \text{ on } \Sigma_\nu \cap (R_3^\circ \cup \delta'' R_3) \\ \Rightarrow \forall\,(x,t) \in R_1 \cap \Sigma_\nu \,:\, |v_x^\nu(x,t)| \leq C \max_{\Sigma_\nu \cap R_3} |v^\nu|.$$

Proof. We must reduce the problem to the situation covered in theorem 8.8. Even though $R_1 \subset R_3$ there need not be room for a μ-rectangle R_2 between them, so that

$$R_1 \subset R_2 \subset R_3$$

and $\delta' R_i \cap \delta' R_{i+1} = \emptyset$, $i = 1, 2$. But R_1 and R_3 are also ν-rectangles for $\nu \geq \mu$, in particular for $\nu = \mu + 1$, and it is certainly possible to choose a $(\mu+1)$-rectangle R_2 between R_1 and R_3. Since

$$L(v^\nu) \equiv 0 \text{ on } \Sigma_\nu \cap (R_3^\circ \cup \delta'' R_3)$$

we certainly have

$$L(v^\nu) \equiv 0 \text{ on } \Sigma_\nu \cap (R_2^\circ \cup \delta'' R_2)$$

and now theorem 8.8 applies.

We now turn to the quite lengthy proof of theorem 8.9. For the sake of notational simplicity as well as generality we shall omit all references to ν.

We need a series of lemmata concerning the behaviour of L.

8.12 Lemma. *If $L(v) = a v_{x\overline{x}} + \frac{1}{2} b (v_x + v_{\overline{x}}) + cv - v_{\overline{t}}$ then*

$$L(fg) = fL(g) + gL(f) - cfg \\ + \left(a + \frac{1}{2}bh\right) f_x g_x + \left(a - \frac{1}{2}bh\right) f_{\overline{x}} g_{\overline{x}} + k f_{\overline{t}} g_{\overline{t}}.$$

Proof. The proof is elementary, but a bit technical and is included for convenience. We need the identities

$$(fg)_x = (E_x f) g_x + f_x g \qquad (fg)_{\overline{x}} = f g_{\overline{x}} + f_{\overline{x}} E_x^{-1} g$$
$$(fg)_{x\overline{x}} = f g_{x\overline{x}} + g f_{x\overline{x}} + f_x g_x + f_{\overline{x}} g_{\overline{x}} \qquad (fg)_{\overline{t}} = f g_{\overline{t}} + f_{\overline{t}} E_t^{-1} g.$$

Thus

$$\begin{aligned}
L(fg) &= a(fg)_{x\overline{x}} + \frac{1}{2}b\left((fg)_x + (fg)_{\overline{x}}\right) + cfg - (fg)_{\overline{t}} \\
&= a\left(fg_{x\overline{x}} + gf_{x\overline{x}} + f_x g_x + f_{\overline{x}} g_{\overline{x}}\right) + \frac{1}{2}b\left((E_x f)g_x + f_x g + fg_{\overline{x}} + f_{\overline{x}} E_x^{-1} g\right) \\
&\qquad + cfg - fg_{\overline{t}} - f_{\overline{t}} E_t^{-1} g \\
&= f\left(ag_{x\overline{x}} + \frac{1}{2}b(g_x + g_{\overline{x}}) + cg - g_{\overline{t}}\right) - \frac{1}{2}bfg_x \\
&\qquad + \left(af_{x\overline{x}} + \frac{1}{2}b(f_x + f_{\overline{x}}) + cf - f_{\overline{t}}\right)g - \frac{1}{2}bf_{\overline{x}}g - cfg + gf_{\overline{t}} \\
&\qquad\qquad + af_{\overline{x}}g_{\overline{x}} + af_x g_x + \frac{1}{2}b(E_x f)g_x + \frac{1}{2}bf_{\overline{x}}E_x^{-1}g - f_{\overline{t}}E_t^{-1}g \\
&= fL(g) + L(f)g - cfg + af_x g_x + af_{\overline{x}}g_{\overline{x}} \\
&\qquad + \frac{1}{2}b(E_x f - f)g_x - \frac{1}{2}bf_{\overline{x}}(g - E_x^{-1}g) + f_{\overline{t}}(g - E_t^{-1}g) \\
&= fL(g) + gL(f) - cfg + \left(a + \frac{1}{2}bh\right)f_x g_x + \left(a - \frac{1}{2}bh\right)f_{\overline{x}}g_{\overline{x}} + kf_{\overline{t}}g_{\overline{t}}
\end{aligned}$$

and we are done.

8.13 Lemma. *If* $L(v) = av_{x\overline{x}} + \frac{1}{2}b(v_x + v_{\overline{x}}) + cv - v_{\overline{t}} = 0$ *then*

$$L(v^2) = -cv^2 + \left(a + \frac{1}{2}bh\right)v_x^2 + \left(a - \frac{1}{2}bh\right)v_{\overline{x}}^2 + kv_{\overline{t}}^2.$$

In particular if $c \leq 0$ *and* $h \leq h_{max}$ *then* $L(v) = 0$ *implies that* $L(v^2) \geq 0$.

Proof. The first part of the lemma is an obvious corollary to the previous lemma: simply insert $v = f = g$. The second part is equally easy: $h \leq h_{max}$ implies that $a \pm \frac{1}{2}bh \geq 0$ at all grid points.

8.14 Lemma. *If* $L(v) = av_{x\overline{x}} + \frac{1}{2}b(v_x + v_{\overline{x}}) + cv - v_{\overline{t}}$ *then*

$$L(v)_x = L(v_x) + a_x v_{xx} + \frac{1}{2}b_x(E_x v_x + v_x) + c_x E_x v.$$

Proof. The proof is immediate, but please use the identity

$$(fg)_x = f_x E_x g + fg_x$$

rather than $(fg)_x = (E_x f)g_x + f_x g$. Then

$$\begin{aligned}
L(v)_x = a_x v_{xx} + \frac{1}{2}b_x(E_x v_x + E_x v_{\overline{x}}) + c_x E_x v \\
+ \underbrace{av_{x\overline{x}x} + \frac{1}{2}b(v_{xx} + v_{x\overline{x}}) + cv_x - v_{\overline{t}x}}_{L(v_x)}
\end{aligned}$$

and we are done.

We now turn to the calculation and estimation of $L(v_x^2 F)$. By lemma 8.12

$$L(Fv_x^2) = L(F)v_x^2 + FL(v_x^2) - cFv_x^2 + \left(a + \frac{1}{2}bh\right) F_x(v_x^2)_x + \left(a - \frac{1}{2}bh\right) F_{\overline{x}}(v_x^2)_{\overline{x}} + kF_{\overline{t}}(v_x^2)_{\overline{t}}$$

and similarly

$$L(v_x^2) = 2v_x L(v_x) - cv_x^2 + \left(a + \frac{1}{2}bh\right) v_{xx}^2 + \left(a - \frac{1}{2}bh\right) v_{x\overline{x}}^2 + kv_{x\overline{t}}^2.$$

Now unlike the case of the heat equation $L(v_x)$ is probably not zero, but using $L(v) = 0$ along with lemma 8.14 we find that

$$L(v_x^2) = -2v_x \left(a_x v_{xx} + \frac{1}{2}b_x(E_x v_x + v_x) + c_x E_x v\right) - cv_x^2 + \left(a + \frac{1}{2}bh\right) v_{xx}^2 + \left(a - \frac{1}{2}bh\right) v_{x\overline{x}}^2 + kv_{x\overline{t}}^2.$$

As in chapter 7 we shall need the identities

$$\begin{aligned} (v_x^2)_x &= (E_x v_x)v_{xx} + v_x v_{xx} = (v_x + E_x v_x)v_{xx} \\ (v_x^2)_{\overline{x}} &= E_x^{-1}(v_x^2)_x = (v_x + E_x^{-1} v_x)v_{x\overline{x}} \\ k(v_x^2)_{\overline{t}} &= v_x^2 - E_t^{-1} v_x^2. \end{aligned}$$

Now consider the entire expression for $L(Fv_x^2)$:

$$\begin{aligned} L(Fv_x^2) &= L(F)v_x^2 - cFv_x^2 + FL(v_x^2) \\ &\quad + \left(a + \frac{1}{2}bh\right) F_x(v_x^2)_x + \left(a - \frac{1}{2}bh\right) F_{\overline{x}}(v_x^2)_{\overline{x}} + kF_{\overline{t}}(v_x^2)_{\overline{t}} \\ &= L(F)v_x^2 - cFv_x^2 + F\left(a + \frac{1}{2}bh\right) v_{xx}^2 + F\left(a - \frac{1}{2}bh\right) v_{x\overline{x}}^2 + kFv_{x\overline{t}}^2 - cFv_x^2 \\ &\quad - 2Fv_x\left(a_x v_{xx} + \frac{1}{2}b_x(v_x + E_x v_x) + c_x E_x v\right) \\ &\quad + \left(a + \frac{1}{2}bh\right) F_x(v_x + E_x v_x)v_{xx} + \left(a - \frac{1}{2}bh\right) F_{\overline{x}}(v_x + E_x^{-1} v_x)v_{x\overline{x}} \\ &\quad + F_{\overline{t}}\left(v_x^2 - E_t^{-1} v_x^2\right). \end{aligned}$$

Focus on the terms involving v_x and $v_{x\overline{x}}$. They are

$$F\left(a - \frac{1}{2}bh\right) v_{x\overline{x}}^2 + \left(a - \frac{1}{2}bh\right) F_{\overline{x}}(v_x + E_x^{-1} v_x)v_{x\overline{x}}$$

and since

$$F_{\overline{x}} = -th(2x-h)^2 + 2t(2x-h)(x^2-\alpha^2)$$

we concentrate on the term containing the factor $(x^2-\alpha^2)$ and complete the squares. Thus

$$\begin{aligned} F\left(a-\frac{1}{2}bh\right)v_{x\overline{x}}^2 + \left(a-\frac{1}{2}bh\right)\left[2t(2x-h)(x^2-\alpha^2)\right](v_x+E_x^{-1}v_x)v_{x\overline{x}} \\ = \left(a-\frac{1}{2}bh\right)t\left((x^2-\alpha^2)v_{x\overline{x}} + (2x-h)(v_x+E_x^{-1}v_x)\right)^2 \\ - \left(a-\frac{1}{2}bh\right)t(2x-h)^2(v_x+E_x^{-1}v_x)^2 \\ \geq -\left(a-\frac{1}{2}bh\right)t(2x-h)^2\left(2v_x^2+2E_x^{-1}v_x^2\right). \end{aligned}$$

Consider the map

$$(x,t,h) \to -\left(a-\frac{1}{2}bh\right)t(2x-h)^2.$$

If we are given $h_0 > 0$ then this map is bounded by

$$\left(\|a\|_\infty + \frac{1}{2}\|b\|_\infty h_0\right)T(2\alpha+h_0)^2$$

for all $(x,t,h) \in R \times [0,h_0]$.

Consider similarly the terms involving v_x, v_{xx} and the nice part of F_x containing the factor $(x^2-\alpha^2)$, that is

$$\begin{aligned} \underbrace{t(x^2-\alpha^2)^2}_{F}\left(\left(a+\frac{1}{2}bh\right)v_{xx}^2 - 2v_x a_x v_{xx}\right) \\ + \left(a+\frac{1}{2}bh\right)\underbrace{\left[2t(2x+h)(x^2-\alpha^2)\right]}_{\text{only part of } F_x}(v_x+E_xv_x)v_{xx}. \end{aligned} \tag{8.3}$$

If $h < h_{max}$ then division by $\left(a+\frac{1}{2}bh\right)$ leaves us with the nasty factor of

$$t(x^2-\alpha^2)^2v_{xx}^2 - 2\frac{ta_x(x^2-\alpha^2)^2}{\left(a+\frac{1}{2}bh\right)}v_xv_{xx} + 2t(2x+h)(x^2-\alpha^2)(v_x+E_xv_x)v_{xx} \tag{8.4}$$

which should be treated as an incomplete square of a sum of three terms involving v_{xx}, v_x and $v_x + E_xv_x$. We must identify the missing terms; add them and subtract them again. The result is

$$t\left\{[(x^2-\alpha^2)v_{xx}]+((2x+h)(v_x+E_xv_x))-\left[\frac{a_x(x^2-\alpha^2)}{a+\frac{1}{2}bh}v_x\right]\right\}^2$$
$$-t\left(\frac{a_x(x^2-\alpha^2)}{a+\frac{1}{2}bh}\right)^2 v_x^2-t(2x+h)^2(v_x+E_xv_x)^2$$
$$+2t(2x+h)\frac{a_x(x^2-\alpha^2)}{a+\frac{1}{2}bh}(v_x+E_xv_x)v_x.$$

The terms $(v_x+E_xv_x)^2$ and $2(v_x+E_xv_x)v_x$ are quite easy to estimate

$$(v_x+E_xv_x)^2\le 2v_x^2+2E_xv_x^2$$

and similarly

$$2|(v_x+E_xv_x)v_x|\le 3v_x^2+E_xv_x^2.$$

In summary: The expression of equation (8.3) is bounded from below by

$$-t\frac{(a_x(x^2-\alpha^2))^2}{a+\frac{1}{2}bh}v_x^2-t\left(a+\frac{1}{2}bh\right)(2x+h)^2(2v_x^2+2E_xv_x^2)$$
$$+t(2x+h)a_x\left(x^2-\alpha^2\right)(3v_x^2+E_xv_x^2).$$

Please note that if we choose $h_0<h_{max}$ then the functions

$$(x,t,h)\to t\frac{[a_x(x^2-\alpha^2)]^2}{a+\frac{1}{2}bh}$$
$$(x,t,h)\to t\left(a+\frac{1}{2}bh\right)(2x+h)^2$$
$$(x,t,h)\to t(2x+h)a_x(x^2-\alpha^2)$$

are bounded on the set $R\times[0,h_0]$. Specific bounds are given by

$$T\frac{\left\|\frac{\partial a}{\partial x}\right\|_\infty^2\alpha^4}{\inf a-\frac{1}{2}\|b\|_\infty h_0} \tag{8.5}$$

$$T\left(\|a\|_\infty+\frac{1}{2}\|b\|_\infty h_0\right)(2\alpha+h_0)^2 \tag{8.6}$$

$$T(2\alpha+h_0)\left\|\frac{\partial a}{\partial x}\right\|_\infty\alpha^2 \tag{8.7}$$

where $\left\|\frac{\partial a}{\partial x}\right\|_\infty$ is the supremum-norm of the *smooth* derivative $\frac{\partial a}{\partial x}$.

We still have to deal with the remaining parts of F_x and $F_{\overline{x}}$. They amount to

$$\left(a+\frac{1}{2}bh\right)\underbrace{th(2x+h)^2}_{\text{from } F_x}(v_x+E_xv_x)v_{xx}-\left(a-\frac{1}{2}bh\right)\underbrace{th(2x-h)^2}_{\text{from } F_{\overline{x}}}(v_x+E_x^{-1})v_{x\overline{x}}.$$

The key is to observe that

$$h(v_x + E_x v_x)v_{xx} = h(v_x^2)_x = E_x v_x^2 - v_x^2$$
$$h(v_x + E_x^{-1} v_x)v_{x\overline{x}} = h(v_x^2)_{\overline{x}} = v_x^2 - E_x^{-1} v_x^2$$

which reduces the expression to one of the desired type, i.e. a function which is bounded on $R \times (0, h_0]$ times a sum of shifted squares of v_x.

There are still several terms left in the complete expression for $L(Fv_x^2)$. The terms

$$-cFv_x^2, \quad kF(v_{x\overline{t}})^2, \quad F_{\overline{t}} v_x^2$$

are all positive and may be discarded, because we are attempting to establish a lower bound. The terms

$$L(F)v_x^2, \quad -F_{\overline{t}} E_t^{-1} v_x^2$$

are of the desired type and can be dominated. Given $h_0 > 0$ the factor $L(F)$ will be bounded for $(x, t, h) \in R \times (0, h_0]$. To see this we recall that $F_{x\overline{x}}$ and $\frac{1}{2}(F_x + F_{\overline{x}})$ are bounded by the values of $\frac{\partial^2 F}{\partial x^2}$ and $\frac{\partial F}{\partial x}$, respectively, on the slightly larger rectangle of

$$[-\alpha - h_0, \alpha + h_0] \times [0, T].$$

Similarly $F_{\overline{t}}$ is bounded on $R \times (0, h_0]$ by the values of $\frac{\partial F}{\partial t}$ on the rectangle

$$[-\alpha, \alpha] \times [-h_0, T].$$

Since $a, b, c,$ and f are bounded on $\overline{\Omega}$ we discover that $L(F)$ is bounded on $R \times (0, h_0]$.

Finally we have the term

$$-2v_x \frac{1}{2} b_x (v_x + E_x v_x) = -b_x v_x^2 - b_x v_x E_x v_x \geq -b_x v_x^2 - |b_x| \left(\frac{1}{2} v_x^2 + \frac{1}{2} E_x v_x^2 \right)$$

and

$$-2v_x c_x E_x v \geq -|c_x| \left(v_x^2 + E_x v_x^2 \right).$$

The *divided differences* b_x, c_x are of course bounded independently of h by the supremum-norm of the corresponding derivatives, i.e. $\frac{\partial b}{\partial x}$ and $\frac{\partial c}{\partial x}$.

Thus both expressions are bounded from below in the desired fashion.

And then there are no more terms ! An examination of the previous argument reveals that we merely need to add multiples of

$$v_x^2, \quad E_x v_x^2, \quad E_t^{-1} v_x^2, \quad E_x^{-1} v_x^2$$

in order to secure control over $L(Fv_x^2)$ and achieve our goal of $L(z) \geq 0$. But such factors are found in abundance within $L(w)$. Observe how lemma 8.13 gives us

$$L(v^2) = -cv^2 + \left(a + \frac{1}{2} bh \right) v_x^2 + \left(a - \frac{1}{2} bh \right) v_{\overline{x}}^2 + kv_{\overline{t}}^2$$

so that

$$\begin{aligned}
L(E_x^{-1}v^2) &= -cE_x^{-1}v^2 + \underline{\left(a+\frac{1}{2}bh\right)E_x^{-1}v_x^2} + \left(a-\frac{1}{2}bh\right)E_x^{-1}v_{\bar{x}}^2 + kE_x^{-1}v_{\bar{t}}^2 \\
L(E_xv^2) &= \underline{-cE_xv^2} + \left(a+\frac{1}{2}bh\right)E_xv_x^2 + \underline{\left(a-\frac{1}{2}bh\right)E_xv_{\bar{x}}^2} + kE_xv_{\bar{t}}^2 \\
L(E_t^{-1}v^2) &= -cE_t^{-1}v^2 + \underline{\left(a+\frac{1}{2}bh\right)E_t^{-1}v_x^2} + \left(a-\frac{1}{2}bh\right)E_t^{-1}v_{\bar{x}}^2 + kE_t^{-1}v_{\bar{t}}^2.
\end{aligned}$$

The underlined terms are all that we need; the rest may be discarded.

We are almost done now. We choose $h_0 < h_{max}$ and limit ourselves to

$$h \in (0, h_0].$$

Then the previous calculations are valid, in particular we do not divide by zero in equation (8.4) and all our explicit bounds are finite, fx. equation (8.5). But equally important is the fact that

$$L(w) \geq \left(\inf a - \frac{1}{2}\|b\|_\infty h_0\right)\left(E_x^{-1}v_x^2 + E_xv_{\bar{x}}^2 + E_t^{-1}v_x^2\right) - (\sup c)E_xv^2.$$

Since $\inf a - \frac{1}{2}\|b\|_\infty h_0 > 0$ and $-\sup c > 0$ we can choose a $C_0 > 0$ such that $L(z) \geq 0$ for all $C \geq C_0$. C_0 can be calculated in terms of h_0, the rectangle R and the relevant functions $a, b, c,$ and F. It is important to realize that C_0 does not depend on v nor on $h \in (0, h_0)$. This finally completes the proof of theorem 8.9.

We must now attempt to bound v_{xx}^ν, v_{xxx}^ν and v_{xxxx}^ν independently of ν on every compact set. As in the case of the heat equation the difference equation itself can be used to express the remaining divided differences

$$v_{\bar{t}}^\nu, v_{\bar{t}x}^\nu, v_{\bar{t}xx}^\nu, v_{\bar{t}\bar{t}}^\nu$$

in terms of $v^\nu, v_x^\nu, v_{xx}^\nu, v_{xxx}^\nu$ and v_{xxxx}^ν and although the calculations are a bit more complicated they are still elementary and will be postponed.

Consider the smooth problem of providing an a priori estimate for $\frac{\partial^2 u}{\partial x^2}$ on a closed rectangle $R \subset \Omega \cup \delta''\Omega$. We may assume that R lies along the final line of $t = T$, because otherwise we simply replace T with $\max\limits_{(x,t)\in R} t$.

If u is a smooth function such that

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + cu$$

then

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) = a\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial a}{\partial x} + b\right)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial b}{\partial x} + c\right)\frac{\partial u}{\partial x} + \frac{\partial c}{\partial x}u$$

and we see that $\frac{\partial u}{\partial x}$ satisfies an equation which is similar to the equation satisfied by u. There are two differences: The creation term has been modified to $\frac{\partial b}{\partial x}+c$, but since we have substantial control over c we may assume that not only is $c<0$ but also $\frac{\partial b}{\partial x}+c<0$. But because of the inhomogeneous term of $\frac{\partial c}{\partial x}u$ we can not apply the techniques of chapter 6 and estimate $\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$ relative to $\frac{\partial u}{\partial x}$. Instead we use the following trick. Let $R_1=R$ and find closed rectangles R_2, R_3, R_4 such that

$$R_1 \subset R_2 \subset R_3 \subset R_4 \subset (\Omega \cup \delta''\Omega)$$

and $\delta' R_i \cap \delta' R_{i+1}=\emptyset$, for $i=1,2,3$. Obtain an estimate of u_x relative to u on the compact set of R_4. Design a smooth function $\phi:\mathbb{R}^2\to[0,1]$ such that $\phi\equiv 1$ on R_2 and $\phi\equiv 0$ outside of R_3. The important thing is that ϕ is infinitely differentiable with bounded derivatives and that ϕ vanishes completely before we run out of information on the size of u_x.

Now consider the differential equation

$$\frac{\partial w}{\partial t}=a\frac{\partial^2 w}{\partial x^2}+\left(\frac{\partial a}{\partial x}+b\right)\frac{\partial w}{\partial x}+\left(\frac{\partial b}{\partial x}+c\right)w+\phi\frac{\partial c}{\partial x}u$$

which is quite similar to the equation satisfied by $\frac{\partial u}{\partial x}$.

Recall that a, b, and c are assumed to be defined not only within Ω but on the entire strip S_T and also that $\inf_{S_T} a>0$. While u is only defined within $\overline{\Omega}$ the combination $\phi\frac{\partial c}{\partial x}u$ makes sense for all $(x,t)\in S_T$. We may therefore consider the pure initial value problem of

$$\frac{\partial w}{\partial t}=a\frac{\partial^2 w}{\partial x^2}+\left(\frac{\partial a}{\partial x}+b\right)\frac{\partial w}{\partial x}+\left(\frac{\partial b}{\partial x}+c\right)w+\phi\frac{\partial c}{\partial x}u, \quad (x,t)\in S_T$$

with $w\equiv 0$ on the initial line of $t=0$. The function w is easy to estimate. Please note that the inhomogeneous term of $\phi\frac{\partial c}{\partial x}u$ is bounded and

$$\left|\phi\frac{\partial c}{\partial x}u\right|\le\left\|\frac{\partial c}{\partial x}\right\|_\infty\|g\|_\infty, \quad (x,t)\in S_T$$

because $|\phi|\le 1$ and ϕ vanishes before we leave Ω, within which we have the estimate of $|u|\le\|g\|_\infty$. Then by the smooth maximum principle

$$|w(x,t)|\le T\left\|\frac{\partial c}{\partial x}\right\|_\infty\|g\|_\infty, \quad (x,t)\in S_T \tag{8.8}$$

since by general assumption even $\frac{\partial b}{\partial x}+c<0$.

The differential equation satisfied by w_x is

$$\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial x}\right)=a\frac{\partial^2}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)+\left(2\frac{\partial a}{\partial x}+b\right)\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)+\left(\frac{\partial^2 a}{\partial x^2}+2\frac{\partial b}{\partial x}+c\right)\frac{\partial w}{\partial x}$$
$$+\left(\frac{\partial^2 b}{\partial x^2}+\frac{\partial c}{\partial x}\right)w+\frac{\partial}{\partial x}\left(\phi\frac{\partial c}{\partial x}u\right), \quad (x,t)\in S_T$$

with $w_x \equiv 0$ on the initial line of $t = t_0$. Thus we may estimate the size of w_x provided that we can estimate the *inhomogeneous* term of

$$\left(\frac{\partial^2 b}{\partial x^2} + \frac{\partial c}{\partial x}\right) w + \frac{\partial}{\partial x}\left(\phi \frac{\partial c}{\partial x} u\right).$$

The term $\left(\frac{\partial^2 b}{\partial x^2} + \frac{\partial c}{\partial x}\right) w$ is easily dominated using our global estimate (8.8) for w, while

$$\frac{\partial}{\partial x}\left(\phi \frac{\partial c}{\partial x} u\right) = \frac{\partial \phi}{\partial x}\frac{\partial c}{\partial x} u + \phi \frac{\partial^2 c}{\partial x^2} u + \phi \frac{\partial c}{\partial x}\frac{\partial u}{\partial x}$$

is zero outside of the rectangle R_3 which lies well within R_4 on which we have estimates of $\frac{\partial u}{\partial x}$ as well as u.

Consider also the initial-boundary value problem of

$$\frac{\partial \omega}{\partial t} = a \frac{\partial^2 \omega}{\partial x^2} + \left(\frac{\partial a}{\partial x} + b\right)\frac{\partial \omega}{\partial x} + \left(\frac{\partial b}{\partial x} + c\right)\omega + (1-\phi)\frac{\partial c}{\partial x} u$$

for $(x, t) \in R_2^\circ \cup \delta'' R_2$ and $\omega = u_x - w$ on the parabolic boundary of R_2. Let us suppose that this problem actually has a solution $\omega \in C(R_2) \cap C^\infty(R_2^\circ \cup \delta'' R)$. Then

$$\frac{\partial}{\partial t}(w + \omega) = a \frac{\partial^2}{\partial x^2}(w + \omega) + \left(\frac{\partial a}{\partial x} + b\right)\frac{\partial}{\partial x}(w + \omega)$$
$$+ \left(\frac{\partial b}{\partial x} + c\right)(w + \omega) + (\phi + (1 - \phi))\frac{\partial c}{\partial x} u$$

within $R_2^\circ \cup \delta'' R$ and $w + \omega = \frac{\partial u}{\partial x}$ on the parabolic boundary of R_2. By the uniqueness theorem 5.7 $w + \omega$ would be identical to $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2} = \frac{\partial w}{\partial x} + \frac{\partial \omega}{\partial x}$. Returning to the equation for ω we discover that there is in fact no inhomogeneous term because $\phi \equiv 1$ on R_2 and by general assumption $\sup(\frac{\partial b}{\partial x} + c)$ will be strictly negative. Thus by theorem 6.3 we may use estimate $\frac{\partial \omega}{\partial x}$ relative to ω on the smaller rectangle R_1

$$\left\|\frac{\partial \omega}{\partial x}\right\|_{R_1,\infty} \le C_{R_1} \|\omega\|_{R_2,\infty}.$$

Furthermore ω will be bounded by its values on the parabolic boundary of R_2. But these are merely $\omega = \frac{\partial u}{\partial x} - w$ which we can estimate. It is therefore possible to provide an a priori estimate for $\frac{\partial^2 u}{\partial x^2}$ on $R = R_1$.

This exercise can be carried out for $\frac{\partial^3 u}{\partial x^3}, \frac{\partial^4 u}{\partial x^4}$ and so on. Let us instead demonstrate how we may apply these ideas to the discrete case.

Our difference equation is a special case of the following equation

$$v_{\overline{t}} = a v_{x\overline{x}} + b_1 v_x + b_2 v_{\overline{x}} + c_1 v + c_2 E_x^{-1} v + f. \tag{8.9}$$

Normally we would have $b_1 = b_2 = \frac{1}{2}b, c_1 = c, c_2 = 0$ and perhaps $f = 0$. As in the chapter on the pure initial value problem we consider this scheme because it is invariant under δ_x in the sense that if v satisfies 8.9 then v_x satisfies

$$(v_x)_{\overline{t}} = E_x a (v_x)_{x\overline{x}} + (E_x b_1)(v_x)_x + (E_x b_2 + a_x)(v_x)_{\overline{x}} + (E_x c_1 + b_{1x})(v_x) \\ + (E_x c_2 + b_{2x})E_x^{-1} v_x + (f_x + c_{1x} v + c_{2x} E_x^{-1} v). \quad (8.10)$$

Writing

$$\tilde{a} = E_x a \qquad \tilde{b}_1 = E_x b_1 \qquad \tilde{b}_2 = E_x b_2 + a_x,$$
$$\tilde{c}_1 = E_x c_1 + b_{1x} \qquad \tilde{c}_2 = E_x c_2 + b_{2x} \qquad \tilde{f} = f_x + c_{1x} v + c_{2x} E_x^{-1} v$$

we see that

$$(v_x)_{\overline{t}} = \tilde{a}(v_x)_{x\overline{x}} + \tilde{b}_1 (v_x)_x + \tilde{b}_2 (v_x)_{\overline{x}} + \tilde{c}_1 (v_x) + \tilde{c}_2 E_x^{-1}(v_x) + \tilde{f}$$

and it is apparent that v_x satisfies an equation which is of the same type as the one satisfied by v. This is why we say that equation (8.9) is invariant under δ_x.

8.15 Remark. At this juncture we ask the reader to take note of the fact that

$$\tilde{c_1} + \tilde{c_2} = E_x(c_1 + c_2) + b_{1x} + b_{2x}.$$

Thus we may exercise some measure of control over $\tilde{c_1} + \tilde{c_2}$ by adjusting $c_1 + c_2$.

We claim that if

$$v_{\overline{t}} = a v_{x\overline{x}} + b_1 v_x + b_2 v_{\overline{x}} + c_1 v + c_2 E_x^{-1} v$$

with $\sup(c_1 + c_2) < 0$ then it is possible to estimate v_x relative to v on every compact set of Ω. We have already done so in the special case of $b_1 = b_2 = \frac{1}{2}b$ and $c_2 = 0$ and we shall prove the generalisation shortly.

Let R be a closed μ-rectangle. We may assume that R lies on the final line of $t = T$. Inspired by our analysis of the smooth problem we pick four closed ν-rectangles $(\nu \geq \mu)$ R_1, R_2, R_3 and R_4 such that

$$R = R_1 \subset R_2 \subset R_3 \subset R_4 \subset \Omega \cup \delta'' \Omega$$

and $\delta' R_i \cap \delta' R_{i+1} = \emptyset$ for $i = 1, 2, 3$. We obtain an estimate of v_x relative to v on the compact set of R_4 which is independent of ν. Let ϕ be the corresponding cut-off function, such that $\phi \equiv 1$ on R_2 and $\phi \equiv 0$ outside of R_3.

Consider the initial value problem of

$$w_{\overline{t}} = (E_x a) w_{x\overline{x}} + (E_x b_1) w_x + (E_x b_2 + a_x) w_{\overline{x}} + (E_x c_1 + b_{1x}) w \\ + (E_x c_2 + b_{2x}) E_x^{-1} w + \phi \left[(f_x + c_{1x} v + c_{2x} E_x^{-1} v) \right]$$

on the entire grid $\Sigma_{h,k}$ along with the initial condition of $w \equiv 0$. We claim that it is possible to solve this implicit equation and to obtain estimates for w in terms of the size of the inhomogeneous term of

$$\phi \left[f_x + c_{1x} v + c_{2x} E_x^{-1} v \right]$$

which we can estimate. We also claim that it is possible to estimate w_x by studying the equation satisfied by w_x.

Consider also the initial-boundary value problem of

$$\begin{aligned} \omega_{\overline{t}} = (E_x a)\omega_{x\overline{x}} + (E_x b_1)\omega_x + (E_x b_2 + a_x)\omega_{\overline{x}} + (E_x c_1 + b_{1x})\omega \\ + (E_x c_2 + b_{2x}) E_x^{-1} \omega + (1 - \phi) \left[(f_x + c_{1x} v + c_{2x} E_x^{-1} v) \right] \end{aligned}$$

on $\Sigma_{h,k} \cap (R_2 - \delta' R_2)$ with $\omega = v_x - w$ on the parabolic boundary nodes of R_2. Since $\phi \equiv 1$ on R_2 the inhomogeneous term vanishes and we have already claimed that it is possible to estimate ω_x relative to ω on R_1 because by general assumption even

$$\sup(E_x(c_1 + c_2) + b_{1x} + b_{2x}) < 0.$$

A uniqueness theorem would then guarantee that $v = w + \omega$ and an estimate of $v_{xx} = w_x + \omega_x$ would then be at hand.

We now turn to the task of proving all our claims regarding the generalised implicit finite difference operator L given by

$$L(v) = a v_{x\overline{x}} + b_1 v_x + b_2 v_{\overline{x}} + c_1 v + c_2 E_x^{-1} v - v_{\overline{t}}. \tag{8.11}$$

We begin by establishing a maximum principle for the corresponding initial-boundary value problem.

8.16 Theorem.

$$\begin{aligned} &\forall \lambda_0 > 0 \, \exists h_0, k_0 > 0 \, \forall h \in (0, h_0] \\ &\qquad k \in [\lambda_0 h^2, k_0] \Rightarrow |v_j^n| \le \left(\frac{1}{1 - Ck} \right)^n \left(\max_{\delta' \Omega_{h,k}} |v| + nk \max_{\Omega_{h,k}} |f| \right) \end{aligned}$$

for all solutions $v : \overline{\Omega}_{h,k} \to \mathbb{R}$ *of the difference equation*

$$v_{\overline{t}} = a v_{x\overline{x}} + b_1 v_x + b_2 v_{\overline{x}} + c_1 v + c_2 E_x^{-1} v + f$$

within $\Omega_{h,k}$ *and* $v = g_{h,k}$ *on* $\delta' \Omega_{h,k}$. *As usual*

$$C = \max\{0, \sup(c_1 + c_2)\}.$$

Proof. Please note that the curious condition of $k \ge \lambda_0 h^2$ forces $\lambda = k/h^2 \ge \lambda_0$. The difference equation is equivalent to

$$\begin{aligned} &(1 + 2\lambda a + (b_1 - b_2)\lambda h - c_1 k) \, v_j^n \\ &\qquad = (\lambda a + \lambda b_1 h) v_{j+1}^n + (\lambda a - \lambda b_2 h + c_2 k) v_{j-1}^n + k f + v_j^{n-1}. \end{aligned}$$

By the triangle inequality

$$\begin{aligned}&|1 + 2\lambda a + (b_1 - b_2)\lambda h - c_1 k|\,|v_j^n| \\ &\qquad \le |\lambda a + \lambda b_1 h||v_{j+1}^n| + |\lambda a - \lambda b_2 h + c_2 k||v_{j-1}^n| + k|f| + |v_j^{n-1}|. \quad (8.12)\end{aligned}$$

Now if $\|v^n\| = |v_j^n|$ at an *internal* node then

$$\begin{aligned}&|1 + 2\lambda a + (b_1 - b_2)\lambda h - c_1 k|\,\|v^n\| \\ &\qquad \le |\lambda a + \lambda b_1 h|\|v^n\| + |\lambda a - \lambda b_2 h + c_2 k|\|v^n\| + k \max_{\overline{\Omega}_{h,k}} |f| + \|v^{n-1}\|.\end{aligned}$$

Please note that we would experience a substantial cancellation of terms if the leading coefficients were sure to be positive. To this end we note that

$$1 - c_1 k \ge 1 - \|c_1\|_\infty k \ge 0$$

if $k \le \|c_1\|_\infty^{-1}$ and

$$2a + (b_1 - b_2)h \ge 2 \inf a - \|b_1 - b_2\|_\infty h \ge 0$$

if $h \le \frac{2 \inf a}{\|b_1 - b_2\|_\infty}$ such that

$$1 + 2\lambda a + (b_1 - b_2)\lambda h - c_1 k \ge 0$$

if both conditions are met. The second coefficient

$$\lambda a + \lambda b_1 h \ge \lambda(\inf a - \|b_1\|_\infty h) \ge 0$$

if $h \le \frac{\inf a}{\|b_1\|_\infty}$ while the final coefficient of

$$\begin{aligned}&\lambda a - \lambda b_2 h + c_2 k \ge \lambda_0(a - b_2 h) + c_2 k \\ &\qquad \ge \lambda_0(\inf a - \|b_2\|_\infty h) - \|c_2\|_\infty k \ge \frac{1}{2}\lambda_0 \inf a - \|c_2\|_\infty k \ge 0\end{aligned}$$

provided we choose $h \le \frac{\inf a}{2\|b_2\|_\infty}$ and $k \le \frac{\inf a}{2\|c_2\|_\infty}$. Now define k_0 by

$$k_0 = \min\left\{\|c_1\|_\infty^{-1}, \frac{\inf a}{2\|c_2\|_\infty}, \frac{1}{2\|c_1 + c_2\|_\infty}\right\}$$

and h_0 by

$$h_0 = \min\left\{\frac{2\inf a}{\|b_1 - b_2\|_\infty}, \frac{\inf a}{\|b_1\|_\infty}, \frac{\inf a}{2\|b_2\|_\infty}, \sqrt{\frac{k_0}{\lambda_0}}\right\}.$$

Please note that we have added two extra conditions that will be needed shortly.

Now pick $h \le h_0$ and since $h \le \sqrt{\frac{k_0}{\lambda_0}}$ it is actually possible to pick $k \in [\lambda_0 h^2, k_0]$. Returning to the basic inequality of (8.12) we discover that the coefficients are positive and we may conclude that if $\|v^n\| = |v_j^n|$ at an internal node then

$$(1-(c_1+c_2)k)\|v^n\| \le \|v^{n-1}\| + k\max_{\overline{\Omega}_{h,k}} |f|.$$

Choosing $C = \max\{0, \sup(c_1+c_2)\}$ gives

$$\|v^n\| \le \frac{1}{1-Ck}\left(\|v^{n-1}\| + k\max_{\overline{\Omega}_{h,k}} |f|\right)$$

since $1 - Ck \ge \frac{1}{2} > 0$ by the choice of k.

Now let us assume that

$$\|v^n\| \le \left(\frac{1}{1-Ck}\right)^n \left(\max_{\delta'\Omega_{h,k}} |v| + nk\max_{\Omega_{h,k}} |f|\right)$$

for some particular value of n. The inequality is certainly satisfied for $n = 0$ and let us consider $\|v^{n+1}\|$. There are two distinct possibilities: either $|v_j^{n+1}|$ assumes the value $\|v^{n+1}\|$ at an internal node or at a parabolic boundary node of the $(n+1)$'st time step. If $|v_j^n| = \|v^{n+1}\|$ at a parabolic boundary node, then certainly

$$\|v^{n+1}\| \le \max_{\delta'\Omega_{h,k}} |v| \le \left(\frac{1}{1-Ck}\right)^{n+1} \left(\max_{\delta'\Omega_{h,k}} |v| + (n+1)k\max_{\Omega_{h,k}} |f|\right).$$

If on the other hand $|v_j^n| = \|v^{n+1}\|$ at an internal node then the previous analysis applies and by the induction hypothesis

$$\begin{aligned}
\|v^{n+1}\| &\le \frac{1}{1-Ck}\left(\|v^n\| + k\max_{\overline{\Omega}_{h,k}} |f|\right) \\
&\le \frac{1}{1-Ck}\left[\left(\frac{1}{1-Ck}\right)^n \left(\max_{\delta'\Omega_{h,k}} |v| + nk\max_{\Omega_{h,k}} |f|\right)\right] + \frac{k}{1-Ck}\max_{\Omega_{h,k}} |f| \\
&\le \left(\frac{1}{1-Ck}\right)^{n+1} \left(\max_{\delta'\Omega_{h,k}} |v| + (n+1)k\max_{\Omega_{h,k}} |f|\right)
\end{aligned}$$

and we are done with the proof of this theorem.

8.17 Remark. This result is hardly surprising in view of the corresponding result for the explicit method on a rectangle and the smooth maximum principle of chapter 5.

The following corollary is immediate.

8.18 Corollary. *For all $\lambda_0 > 0$ there exists $h_0, k_0 > 0$ such that for all $h \in (0, h_0]$ and $k \in [\lambda_0 h^2, k_0]$ the difference equation*

$$v_{\overline{t}} = a v_{x\overline{x}} + b_1 v_x + b_2 v_{\overline{x}} + c_1 v + c_2 E_x^{-1} v + f$$

has precisely one solution such that $v = g_{h,k}$ at the parabolic boundary nodes.

Proof. By the previous theorem the difference equation has at most one solution. The difference equation is merely a finite system of linear equations. The unknowns are the values of v at the internal nodes and there is an equal number of equations and unknowns. Thus uniqueness implies existence and we are done with the proof of this corollary.

Please note that the solution v is essentially bounded in terms of the boundary values and the values of the driving force f. In particular if $c_1 + c_2 \le 0$ and $f = 0$ then

$$\max_{\overline{\Omega}_{h,k}} |v| \le \max_{\delta'\Omega_{h,k}} |g_{h,k}| \le \|g\|_\infty.$$

The last inequality is independent of the grid.

8.19 Remark. The bound of theorem 8.16 depends on the actual grid and in general we need to replace it with a bound that is independent of the grid. Observe how

$$\max_{\Omega_{h,k}} |f| \le \|f\|_{\overline{\Omega},\infty}$$

and

$$\max_{\delta'\Omega_{h,k}} |v| \le \|g\|_{\delta'\Omega,\infty}$$

while the term of $\left(\frac{1}{1-Ck}\right)^n$ poses a slight problem. By l'Hospital's rule

$$\left(\frac{1}{1-Ck}\right)^n \to e^{Ct} \quad \text{as} \quad n \to \infty \quad \text{with} \quad nk = t$$

but regrettably $\frac{1}{1-Ck} \ge e^{Ck}$. But given $\epsilon > 0$ it is possible to choose $k_\epsilon > 0$ such that for all $0 < k < k_\epsilon$ we have

$$\frac{1}{1-Ck} \le e^{(C+\epsilon)k}.$$

In summary: We shall pick an $\epsilon > 0$ at leisure and estimate

$$|v_j^n| \le e^{(C+\epsilon)T}\left(\|g\|_{\delta'\Omega,\infty} + T\|f\|_{\overline{\Omega},\infty}\right)$$

at all grid points $(jh, nk) \in \overline{\Omega}_{h,k}$ and $k \le k_\epsilon$.

We now consider the pure initial value problem of

$$v_{\overline{t}} = a v_{x\overline{x}} + b_1 v_x + b_2 v_{\overline{x}} + c_1 v + c_2 E_x^{-1} v + f.$$

As always I have found it instructive to begin with the simplest case of

$$v_{\overline{t}} = v_{x\overline{x}}.$$

This equation is equivalent to

$$(1+2\lambda)v_j^n = \lambda v_{j+1}^n + \lambda v_{j-1}^n + v_j^{n-1}$$

or equivalently

$$Av^n = v^{n-1}$$

where $A = A(\lambda)$ is the linear operator defined on $l^\infty(h\mathbb{Z})$ given by

$$(Aw)_j = -\lambda w_{j-1} + (1+2\lambda)w_j - \lambda w_{j+1}.$$

We need to invert A ! A is in fact a bounded operator on $l^\infty(h\mathbb{Z})$ and

$$\|A\| \leq 1 + 4\lambda.$$

To see this we choose $w_j = (-1)^j$ so that $\|w\| = 1$ and $|(Aw)_j| = 1 + 4\lambda$ is maximal. Similarly $\|I - A\| = 4\lambda$ so that A is invertible with

$$A^{-1} = \sum_{k=0}^{\infty} (I - A)^k$$

provided that $\lambda < \frac{1}{4}$ and $\|A^{-1}\| \leq \sum_{k=0}^{\infty} \|(I-A)\|^k = \frac{1}{1-4\lambda}$.

8.20 Remark. The restriction of $\lambda < \frac{1}{4}$ is quite severe and our whole approach is essentially flawed in the sense that in general $A(\lambda)$ will *not* be a small perturbation to the identity operator. However we do not need to develop the theory for *practical* values of λ and thus continue with $\lambda < \frac{1}{4}$.

Knowing that A^{-1} is a bounded operator allows us to find a much better estimate for $\|A^{-1}\|$! Observe how $v^{n-1} \in l^\infty$ now implies that

$$v^n = A^{-1}v^{n-1} \in l^\infty$$

so that $\|v^n\| < \infty$ and

$$\begin{aligned}(1+2\lambda)|v_j^n| &\leq \lambda|v_{j+1}^n| + \lambda|v_{j-1}^n| + |v_j^{n-1}| \\ &\leq \lambda\|v^n\| + \lambda\|v^n\| + \|v^{n-1}\| \\ &\Rightarrow (1+2\lambda)\|v^n\| \leq 2\lambda\|v^n\| + \|v^{n-1}\|\end{aligned}$$

forcing $\|v^n\| \leq \|v^{n-1}\|$ for all choices of v^{n-1}. Thus we discover that $\|A^{-1}\| \leq 1$ but only for $\lambda < \frac{1}{4}$.

We now return to the general problem.

8.21 Theorem. *There exists $\lambda_0 > 0$ and $h_0 > 0$ such that for all $\lambda < \lambda_0$ and $h < h_0$ the difference equation*

$$v_{\overline{t}} = a v_{x\overline{x}} + b_1 v_x + b_2 v_{\overline{x}} + c_1 v + c_2 A_x^{-1} v + f$$

on the grid $\Sigma_h = \{(jh, nk) | j \in \mathbb{Z}, n \in \{0, 1, 2, \dots\}, k = \lambda h^2\}$ *with* $v = g$ *on the initial line has a unique solution.*

Further the solution v *is subject to the estimate*

$$\|v^n\| \le \left(\frac{1}{1 - Ck}\right)^n (\|g\|_\infty + nk\|f\|_\infty)$$

where $C = \max\{0, \sup(c_1 + c_2)\}$.

Proof. The difference equation is equivalent to

$$(1 + 2\lambda a + \lambda(b_1 - b_2)h - c_1 k)v_j^n$$
$$= (\lambda a + \lambda b_1 h)v_{j+1}^n + (\lambda a - \lambda b_2 h + c_2 k)v_{j-1}^n + v_j^{n-1} + kf$$

or $A = A(n, h, k)v^n = v^{n-1} + kf^n$ where

$$(Aw)_j = -(\lambda a + \lambda b_1 h)w_{j+1}^n$$
$$+ (1 + 2\lambda a + \lambda(b_1 - b_2)h - c_1 k)w_j^n - (\lambda a - \lambda b_2 h + c_2 k)w_{j-1}^n.$$

As in the case of the heat equation we estimate A and $I - A$ in order to invert A. Choosing $k = \lambda h^2$ simplifies matters considerably. Observe how the coefficients of A

$$\lambda a + \lambda b_1 h$$
$$1 + 2\lambda a + \lambda(b_1 - b_2)h - c_1 \lambda h^2$$
$$\lambda a - \lambda b_2 h + c_2 \lambda h^2$$

will be positive if $h < \min\{h_1, h_2, h_3\}$ where h_i are the positive solutions to

$$\inf a - \|b_1\|_\infty h_1 = 0$$
$$2 \inf a - \|b_1 - b_2\|_\infty h_2 - \|c\|_\infty h_2^2 = 0$$
$$\inf a - \|b_2\|_\infty h_3 - \|c_2\|_\infty h_3^2 = 0.$$

Choosing $w = (-1)^j$ will now maximize $|(Aw)_j|$ and allow us to estimate $\|A\|$. We discover that

$$|(Aw)_j| = 1 + 4\lambda a + 2\lambda(b_1 - b_2)h + \lambda(c_2 - c_1)h^2$$

and similarly that

$$|(I - A)w_j| = \lambda \left(4a + 2(b_1 - b_2)h + (c_2 - c_1)h^2\right).$$

Thus $\|A\| \le 1 + 6\lambda\|a\|_\infty$ and $\|I - A\| \le 6\lambda\|a\|_\infty$ provided that we choose $h < \min\{h_4, h_5\}$ where

$$2\|b_1 - b_2\|_\infty h_4 = \|a\|_\infty$$
$$\|c_2 - c_1\|_\infty h_5^2 = \|a\|_\infty.$$

Set $\lambda_0 = \frac{1}{6\|a\|_\infty}$ and $h_0 = \min\{h_i | i = 1, 2, 3, 4, 5\}$ and pick $\lambda < \lambda_0, h < h_0$ and $k = \lambda h^2$. Then the previous analysis applies. $A(n, h, k)$ is invertible and the inverse operator $A(n, h, k)^{-1}$ is bounded. Thus we may solve the difference equation recursively as in

$$v^n = A(n, h, k)^{-1}(v^{n-1} + kf^n).$$

Knowing that $v^{n-1} \in l^\infty$ implies that $v^n \in l^\infty$ allows us to develop the desired estimate for $\|v^n\|$. Observe how the difference equation implies that

$$\begin{aligned}&(1 + 2\lambda a_j^n + \lambda(b_{1j}^n - b_{2j}^n)h - c_{1j}^n k)|v_j^n| \\ &\qquad \le (\lambda a_j^n + \lambda b_{1j}^n h)\|v^n\| + (\lambda a_j^n - \lambda b_{2j}^n h + c_{2j}^n k)\|v^n\| + \|v^{n-1}\| + k\|f\|_\infty.\end{aligned}$$

However, there need not be even a single point (jh, nk) such that $|v_j^n| = \|v^n\|$, but given $\|v^n\| > \epsilon > 0$ we may find $j \in \mathbb{Z}$ such that

$$0 < \|v^n\| - \epsilon < |v_j^n|$$

and then we can estimate

$$\begin{aligned}&(1 + 2\lambda a_j^n + \lambda(b_{1j}^n - b_{2j}^n)h - c_{1j}^n k)(\|v^n\| - \epsilon) \\ &\qquad < (\lambda a_j^n + \lambda b_{1j}^n h)\|v^n\| + (\lambda a_j^n - \lambda b_{2j}^n h + c_{2j}^n k)\|v^n\| + \|v^{n-1}\| + k\|f\|_\infty\end{aligned}$$

such that

$$\begin{aligned}\left(1 - (c_{1j}^n + c_{2j}^n)k\right)\|v^n\| &< \|v^{n-1}\| + k\|f\|_\infty \\ &\qquad + \epsilon\left(1 + 2\lambda a_j^n + \lambda(b_{1j}^n - b_{2j}^n)h - c_{1j}^n k\right).\end{aligned}$$

Choosing $C = \max\{0, \sup(c_1 + c_2)\}$ allows us to rid ourselves of j and estimate

$$\begin{aligned}(1 - Ck)\|v^n\| &< \|v^{n-1}\| + k\|f\|_\infty \\ &\qquad + \epsilon\,(1 + 2\lambda \sup a + \lambda\|b_1 - b_2\|_\infty + \|c_1\|_\infty k)\end{aligned}$$

and letting ϵ tend to zero leaves us with the familiar case of

$$(1 - Ck)\|v^n\| \le \|v^{n-1}\| + k\|f\|_\infty$$

which is immediately iterated to yield the desired estimate for $\|v^n\|$ in terms of $\|v^0\| \le \|g\|_\infty$ and $\|f\|_\infty$.

Noticing that our estimate automatically implies uniqueness completes the proof of this theorem.

8.22 Remark. Suppose that we are given the task of estimating not only v, but several of the divided differences of v as well. As an example we consider the problem of estimating v and v_x. By the previous theorem it is possible to select $\lambda_0 > 0, h_0 > 0$ such that the problem is well-posed for $\lambda < \lambda_0$ and $h < h_0$ and estimate

$$\|v^n\| \le \left(\frac{1}{1-Ck}\right)^n (\|g\|_\infty + nk\|f\|_\infty)$$

where $C = \max\{0, \sup(c_1 + c_2)\}$. If $C = 0$ then our bound does not depend on the grid. If $C > 0$ then we proceed as in remark 8.19. Now if v satisfies our difference equation then v_x satisfies the equation

$$(v_x)_{\overline{t}} = (E_x)a(v_x)_{x\overline{x}} + (E_x b_1)(v_x)_x + (E_x b_2 + a_x)(v_x)_{\overline{x}} + (E_x c_1 + b_{1x})(v_x) + (E_x c_2 + b_{2x})E_x^{-1} v_x + (f_x + c_{1x} v + c_{2x} E_x^{-1} v) \quad (8.13)$$

with the initial condition of $v_x^0 = g_x$. Notice that the choice of λ depends exclusively on the leading coefficient which is essentially the same for both equations :

$$\lambda_0 = \frac{1}{6\|a\|_\infty} = \frac{1}{6\|E_x a\|_\infty}.$$

Now in order to apply the previous theorem to v_x we may have to choose a smaller h and we must provide some kind of estimate of the initial condition and the inhomogeneous term. If g is differentiable with a bounded derivative g' then v_x^0 is bounded by $\|g'\|$. The inhomogeneous term of

$$(f_x + c_{1x} v + c_{2x} E_x^{-1} v)$$

can be bounded in terms of $\frac{\partial f}{\partial x}, \frac{\partial c_1}{\partial x}, \frac{\partial c_2}{\partial x}$ and the estimate of v.

We now turn to the problem of estimating v_x relative to v on a closed rectangle $R \subset \Omega$, where v is the solution of

$$v_{\overline{t}} = a v_{x\overline{x}} + b_1 v_x + b_2 v_{\overline{x}} + c_1 v + c_2 E_x^{-1} v$$

in $\Omega_{h,k}$ with $v = g_{h,k}$ on $\delta'\Omega_{h,k}$. We assume that $\sup(c_1 + c_2) < 0$.

The proof is quite similar to the proof for the special case of $b_1 = b_2 = \frac{1}{2} b$ and $c_2 = 0$. We begin with

8.23 Lemma. *If* $L(v) = a v_{x\overline{x}} + b_1 v_x + b_2 v_{\overline{x}} + c_1 v + c_2 E_x^{-1} v - v_{\overline{t}}$ *then*

$$L(fg) = fL(g) + gL(f) + (a + hb_1) f_x g_x + (a - hb_2 + h^2 c_2) f_{\overline{x}} g_{\overline{x}} - c_1 fg - c_2 fg + k f_{\overline{t}} g_{\overline{t}}.$$

Proof. By definition

$$\begin{aligned} L(fg) &= a(fg)_{x\overline{x}} + b_1 (fg)_x + b_2 (fg)_{\overline{x}} + c_1(fg) + c_2 E_x^{-1}(fg) - (fg)_{\overline{t}} \\ &= a(f g_{x\overline{x}} + f_{x\overline{x}} g + f_x g_x + f_{\overline{x}} g_{\overline{x}}) + b_1 \left((E_x f) g_x + f_x g\right) \\ &\quad + b_2 \left(f g_{\overline{x}} + f_{\overline{x}} E_x^{-1}\right) + c_1 fg + c_2 E_x^{-1}(fg) - f g_{\overline{t}} - f_{\overline{t}} E_t^{-1} g. \end{aligned}$$

Aiming towards the term $fL(g)$ we write

$$\begin{aligned} &afg_{x\overline{x}} + fb_2 g_{\overline{x}} + c_1 fg - fg_{\overline{t}} \\ &\quad = f\left(ag_{x\overline{x}} + b_1 g_x + b_2 g_{\overline{x}} + c_1 g + c_2 E_x^{-1} g - g_{\overline{t}}\right) - b_1 f g_x - c_2 f E_x^{-1} g \\ &\quad = fL(g) - b_1 f g_x - c_2 f E_x^{-1} g \end{aligned}$$

and in order to obtain $L(f)g$ we write

$$\begin{aligned} &af_{x\overline{x}} g + b_1 f_x g \\ &= \left(af_{x\overline{x}} + b_1 f_x + b_2 f_{\overline{x}} + c_1 f + c_2 E_x^{-1} f - f_{\overline{t}}\right) g - b_2 f_{\overline{x}} g - c_1 fg - c_2 (E_x^{-1} f) g + f_{\overline{t}} g \\ &= L(f)g - b_2 f_{\overline{x}} g - c_1 fg - c_2 (E_x^{-1} f) g + f_{\overline{t}} g. \end{aligned}$$

Thus

$$\begin{aligned} L(fg) - fL(g) - L(f)g = {} & -b_1 f g_x - c_2 f E_x^{-1} g - b_2 f_{\overline{x}} g - c_1 fg - c_2 (E_x^{-1} f) g + f_{\overline{t}} g \\ & + af_x g_x + af_{\overline{x}} g_{\overline{x}} + b_1 (E_x f) g_x + b_2 f_{\overline{x}} E_x^{-1} g + c_2 E_x^{-1}(fg) - f_{\overline{t}} E_t^{-1} g. \end{aligned}$$

Some of these terms should be paired off immediately as follows

$$\begin{aligned} -b_1 f g_x + b_1 (E_x f) g &= h b_1 f_x g_x \\ f_{\overline{t}} g - f_{\overline{t}} E_t^{-1} g &= k f_{\overline{t}} g_{\overline{t}} \\ b_2 f_{\overline{x}} g - b_2 f_{\overline{x}} E_x^{-1} g &= h b_2 f_{\overline{x}} g_{\overline{x}} \end{aligned}$$

and we are left with the terms

$$-c_2 f E_x^{-1} g - c_2 (E_x^{-1} f) g + c_2 E_x^{-1}(fg).$$

Up to this point the proof has been virtually identical to the proof of lemma 8.12, but we are now feeling the effect of c_2. Observe how

$$-c_2 f E_x^{-1} g = -c_2 f E_x^{-1} g + c_2 fg - c_2 fg = -c_2 fg + hc_2 f g_{\overline{x}}$$

and by symmetry

$$-c_2 (E_x^{-1} f) g = -c_2 fg + hc_2 f_{\overline{x}} g$$

so that

$$\begin{aligned} &- c_2 f E_x^{-1} g - c_2 (E_x^{-1} f) g + c_2 E_x^{-1}(fg) \\ &\qquad = -2c_2 fg + hc_2 \left(f g_{\overline{x}} + f_{\overline{x}} g\right) + c_2 E_x^{-1}(fg). \end{aligned}$$

We use one factor of $-c_2 fg$ as follows

$$-c_2 fg + c_2 E_x^{-1}(fg) = -hc_2 (fg)_{\overline{x}}$$

so that

$$- c_2 f E_x^{-1} g - c_2 (E_x^{-1} f) g + c_2 E_x^{-1}(fg) \\ = -c_2 fg - h\left[(fg)_{\overline{x}} - f g_{\overline{x}} - f_{\overline{x}} g\right].$$

Recalling that $(fg)_{\overline{x}} = f g_{\overline{x}} + f_{\overline{x}} E_x^{-1} g$ leaves us with

$$-c_2 f E_x^{-1} g - c_2 (E_x^{-1} f) g + c_2 E_x^{-1}(fg) = -c_2 fg + h^2 c_2 f_{\overline{x}} g_{\overline{x}}$$

and we are finally done.

The following corollary is immediate.

8.24 Corollary. *If $L(v) = 0$ then*

$$L(v^2) = (a + hb_1) v_x^2 + (a - hb_2 + h^2 c_2) v_{\overline{x}}^2 - (c_1 + c_2) v^2 + k v_{\overline{t}}^2.$$

In particular it is possible to choose h_0 such that for all $h \leq h_0$ we have

$$L(v^2) \geq 0.$$

Proof. Set $f = g = v$ and apply the previous corollary. Pick h_0 such that for all $h \leq h_0$ the coefficients $a + hb_1$ and $a - hb_2 + h^2 c_2$ are non-negative for all possible values of a, b_1, b_2, and c_2.

8.25 Remark. For the sake of symmetry one might want to add an additional term of $c_3 E_x^{-1}$ to L and our "product-rule" lemma 8.23 changes to

$$L(fg) = fL(g) + gL(f) + (a + hb_1 + h^2 c_3) f_x g_x + (a - hb_2 + h^2 c_2) f_{\overline{x}} g_{\overline{x}} \\ - (c_1 + c_2 + c_3) fg + k f_{\overline{t}} g_{\overline{t}}.$$

We do not need this symmetry and we shall not use this symmetric form in the future.

We also need the following lemma.

8.26 Lemma. *Let $\sup(c_1 + c_2) < 0$. For all $\lambda_0 > 0$ there exists $h_0, k_0 > 0$ such that if $h \in (0, h_0), k \in [\lambda_0 h^2, k_0)$ then the inequality*

$$L(v) = v_{\overline{t}} = a v_{x\overline{x}} + b_1 v_x + b_2 v_{\overline{x}} + c_1 v + c_2 E_x^{-1} v - v_{\overline{t}} \geq 0$$

will imply that any global internal maximum will be non-positive. In particular if v has even a single positive value then no such node can be found and the global maximum is achieved only at parabolic boundary nodes.

Proof. Once again the curious condition of $k \geq \lambda_0 h^2$ forces $\lambda \geq \lambda_0$. The inequality is equivalent to

$$(1 + 2\lambda a + \lambda(b_1 - b_2)h - c_1 k) v_j^n \leq (\lambda a + \lambda b_1 h) v_{j+1}^n + (\lambda a - \lambda b_2 h + c_2 k) v_{j-1}^n + v_j^{n-1}.$$

As always we do not proceed before we are certain that the coefficients are positive. Observe how

$$1 - c_1 k \geq 1 - \|c_1\|_\infty k > 0$$

if $k < \frac{1}{\|c_1\|_\infty}$ and

$$2a + (b_1 - b_2)h \geq 2 \inf a - \|b_1 - b_2\|_\infty h > 0$$

if $h < \frac{2 \inf a}{\|b_1 - b_2\|_\infty}$, such that

$$1 + 2\lambda a + \lambda(b_1 - b_2)h - c_1 k > 0$$

if both conditions are satisfied. The second coefficient

$$\lambda a + \lambda b_1 h \geq \lambda(\inf a - \|b_1\|_\infty h) > 0$$

if $h < \frac{\inf a}{\|b_1\|_\infty}$ while the third coefficient

$$\begin{aligned}(\lambda a - \lambda b_2 h + c_2 k) &\geq \lambda_0(a - b_2 h) + c_2 k \\ &\geq \lambda_0(\inf a - \|b_2\|_\infty h) - \|c_2\|_\infty k > 0\end{aligned}$$

if $h < \frac{\inf a}{2\|b_2\|_\infty}$ and $k < \frac{\inf a}{2\|c_2\|_\infty}$. Now define h_0 and k_0 by

$$k_0 = \min\left\{\frac{1}{\|c_1\|_\infty}, \frac{\inf a}{2\|c_2\|_\infty}\right\}$$

and

$$h_0 = \min\left\{\frac{2 \inf a}{\|b_1 - b_2\|_\infty}, \frac{\inf a}{\|b_1\|_\infty}, \frac{\inf a}{2\|b_2\|_\infty}, \sqrt{\frac{k_0}{\lambda_0}}\right\}.$$

Now pick $h \in (0, h_0)$ and since $h < \sqrt{\frac{k_0}{\lambda_0}}$ it is actually possible to pick k in $[\lambda_0 h^2, k_0)$; assume that $L(v) \geq 0$ and that the global maximum $M = \max_{\overline{\Omega}_{h,k}} v_j^n$ is achieved at an internal node $(jh, nk) \in \Omega_{h,k}$. Then $m = v_j^n$ and

$$\begin{aligned}&(1 + 2\lambda a + \lambda(b_1 - b_2)h - c_1 k)M \\ &\qquad \leq (\lambda a + \lambda b_1 h)M + (\lambda a - \lambda b_2 h + c_2 k)M + M \\ &\qquad\qquad \Rightarrow -(c_1 + c_2)M \leq 0.\end{aligned}$$

Since $c_1 + c_2 < 0$ we are left to conclude that $M \leq 0$.

The rest of the proof is trivial. If v assumes even a single positive value then the global maximum must be positive. It can not be achieved at an internal node leaving only the parabolic boundary nodes.

We also need the following lemma.

8.27 Lemma. *If* $L(v) = av_{x\overline{x}} + b_1 v_x + b_2 v_{\overline{x}} + c_1 v + c_2 E_x^{-1} v - v_{\overline{t}}$ *then*

$$L(v_x) = L(v)_x - (a_x v_{xx} + b_{1x} E_x v_x + b_{2x} v_x + c_{1x} E_x v + c_{2x} v_x).$$

Proof. The proof is trivial. We apply $(fg)_x = fg_x + f_x E_x g$ repeatedly

$$\begin{aligned} L(v)_x = av_{xx\overline{x}} + b_1 v_{xx} + b_2 v_{\overline{x}x} + c_1 v_x + c_2 E_x^{-1} v_x - v_{\overline{t}x} \\ + a_x v_{xx} + b_{1x} E_x v_x + b_{2x} v_x + c_{1x} E_x v + c_{2x} v_x \\ = L(v_x) + a_x v_{xx} + b_{1x} E_x v_x + b_{2x} v_x + c_{1x} E_x v + c_{2x} v_x \end{aligned}$$

and we are done.

We are ready for the final analysis. We study Petrowski's function

$$z = F v_x^2 + Cw$$

once again.

The principal problem is the calculation and estimation of $L(Fv_x^2)$. By lemma 8.23

$$\begin{aligned} L(Fv_x^2) = FL(v_x^2) + L(F) v_x^2 + (a + hb_1) F_x (v_x^2)_x \\ + (a - hb_2 + h^2 c_x) F_{\overline{x}} (v_x^2)_{\overline{x}} - (c_1 + c_2) F v_x^2 + k F_{\overline{t}} (v_x^2)_{\overline{t}} \end{aligned}$$

and again by lemma 8.23

$$L(v_x^2) = 2v_x L(v_x) + (a + hb_1) v_x^2 + (a - hb_2 + h^2 c_2) v_{\overline{x}}^2 - (c_1 + c_2) v^2 + k v_{\overline{t}}^2.$$

Applying lemma 8.27 leaves us with

$$\begin{aligned} L(Fv_x^2) = -2Fv_x \left(a_x v_{xx} + b_{1x} E_x v_x + b_{2x} v_x + c_{1x} E_x v + c_{2x} v_x\right) \\ + F\left((a + hb_1) v_x^2 + (a - hb_2 + h^2 c_2) v_{\overline{x}}^2 - (c_1 + c_2) v^2 + k v_{\overline{t}}^2\right) \\ + L(F) v_x^2 + (a + hb_1) F_x (v_x^2)_x + (a - hb_2 + h^2 c_2) F_{\overline{x}} (v_x^2)_{\overline{x}} \\ - (c_1 + c_2) F v_x^2 + k F_{\overline{t}} (v_x^2)_{\overline{t}}. \end{aligned}$$

The task at hand is to bound this expression from below by a sum of terms each consisting of a bounded function multiplied be a suitably shifted square of v_x. The problem is quite similar to the special case of $b_1 = b_2 = \frac{1}{2} b$ and $c_2 = 0$. It is enough to note that $\sup(c_1 + c_2) < 0$ and that it is possible to choose h so small that the coefficients $a + hb_1$ and $a - hb_2 + h^2 c_2$ are positive and bounded away from zero. Then we may proceed in the same fashion as before grouping similar terms together and completing squares.

In summary: It is still possible to find h_0 and $C > 0$ depending exclusively on R and h_0 such that $L(z) \geq 0$ at all grid points within R. By lemma 8.26 z assumes its maximum on the parabolic boundary of R where the cut-off function vanishes by design. Thus we may bound v_x relative to v :

$$\max_{R_1} |v_x| \leq C \max_{R_2} |v|$$

on any pair of μ-rectangles $R_1 \subset R_2$ such that $\delta' R_1 \cap \delta' R_2 = \emptyset$. The constant C does not depend on v or ν. Our relative estimate is valid for all sufficiently large ν.

We are finally ready to generate a solution to our differential equation. We select $\lambda < \lambda_0$ and choose N so large that $k = T/N$ implies that $h = \sqrt{\frac{k}{\lambda}} < h_0$ where λ_0 and h_0 are provided by theorem 8.21. Set $h_\nu = 2^{-\nu} h$ and $k_\nu = \lambda h_\nu^2$ and consider the grids

$$\Sigma_\nu = \{(jh_\nu, nk_\nu) | j, n \in \mathbb{Z}\}$$

and in particular the intersections $\overline{\Omega}_\nu = \overline{\Omega} \cap \Sigma_\nu$. Let $v^\nu : \overline{\Omega}_\nu \to \mathbb{R}$ be *the* solution of the difference equation

$$L(v) = av_{x\overline{x}} + \frac{1}{2} bv_x + \frac{1}{2} bv_{\overline{x}} + cv - v_{\overline{t}} = 0$$

within $\Omega_\nu = \Omega \cap \Sigma_\nu$ and $v^\nu = g_{h_\nu}$ on the parabolic boundary nodes. v^ν is well-defined for h_ν sufficiently small or equivalently $\nu \geq \mu$ for some μ. The exact condition of h_ν is really irrelevant but is given by the proof of lemma 8.16. As we have demonstrated it is possible to bound v^ν_x relative to v^ν on every compact set of Ω at least for ν chosen sufficiently great. In particular it is possible to bound v^ν_x at each point of Ω_ν independently of ν although our bound depends on the point in question. We must also bound $v_{xx}, v_{xxx}, v_{xxxx}$ in a similar fashion. Let $R \subset \Omega$ be a closed μ-rectangle. Our goal is to provide an estimate of v^ν_{xx} on R which is independent of ν. Set $R = R_1$ and find closed μ-rectangles R_2, R_3, R_4 such that

$$R_i \subset R_{i+1}$$

and $\delta' R_i \cap \delta' R_{i+1} = \emptyset$ for $i = 1, 2, 3$. Design a cut-off function $\phi : \mathbb{R}^2 \to [0, 1]$ such that $\phi \equiv 1$ on R_2 and $\phi \equiv 0$ outside of R_3. Obtain an estimate of $\|v_x\|$ relative to $\|v\|$ on the largest rectangle R_4, i.e.

$$\|v_x\|_{R_4,h,k} \leq C_{R_4} \|g\|_\infty.$$

Recall that v_x satisfies the equation

$$(v_x)_{\overline{t}} = (E_x a)(v_x)_{x\overline{x}} + \frac{1}{2}(E_x b)(v_x)_x + \left(\frac{1}{2} E_x b + a_x\right)(v_x)_{\overline{x}} + \left(E_x c + \frac{1}{2} b_x\right)(v_x) + \frac{1}{2} b_x E_x^{-1} v_x + c_x v \quad (8.14)$$

within Ω_ν. Consider the pure initial value problem of

$$w_{\overline{t}} = (E_x a) w_{x\overline{x}} + \frac{1}{2}(E_x b) w_x + \left(\frac{1}{2} E_x b + a_x\right) w_{\overline{x}} + \left(E_x c + \frac{1}{2} b_x\right) w + \frac{1}{2} b_x E_x^{-1} w + \phi c_x v \quad (8.15)$$

with $w \equiv 0$ on the initial line. By assumption a, b, and c are defined not only within $\overline{\Omega}$ but on the the entire strip S_T. The inhomogeneous term of $\phi c_x v$ is defined on the entire grid of Σ_ν because $\phi \equiv 0$ outside of R_3. By theorem 8.21 this problem is well-posed for our choice of λ and since $\left(E_x c + \frac{1}{2} b_x\right) + \frac{1}{2} b_x \leq 0$ we may even estimate

$$|w_j^n| \leq T \|\phi\|_{R_3,\infty} \|c_x\|_{R_3,\infty} \|g\|_{\delta'\Omega,\infty}$$

for $nk \leq T$ and ν sufficiently large.

The divided difference w_x satisfies the equation

$$\begin{aligned}(w_x)_{\overline{t}} = \left(E_x^2 a\right)(w_x)_{x\overline{x}} + \frac{1}{2}\left(E_x^2 b\right)(w_x)_x + \left(\frac{1}{2} E_x^2 b + 2 E_x a_x\right)(w_x)_{\overline{x}} \\ + \left(E_x^2 c + E_x b_x\right) w_x + \left(E_x b_x + a_{xx}\right) E_x^{-1} w_x \\ + \left[(\phi c_x v)_x + (E_x c_x + \frac{1}{2} b_{xx}) w + \frac{1}{2} b_{xx} E_x^{-1} w\right]\end{aligned}$$

with $w_x \equiv 0$ on the initial line. By general assumption even

$$(E_x^2 c + E_x b_x) + (E_x b_x + a_{xx}) \leq 0$$

and again by theorem 8.21 we may estimate

$$|w_x| \leq T \times \text{maximum of inhomogeneous term.}$$

The inhomogeneous term

$$\left[(\phi c_x v)_x + (E_x c_x + \frac{1}{2} b_{xx}) w + \frac{1}{2} b_{xx} E_x^{-1} w\right]$$

is easy to estimate. We have a global estimate of w and $(\phi c_x v)_x$ is certainly zero outside R_4 since ϕ vanishes outside R_3. Within R_4 we write $(\phi c_x v)_x$ in terms of v and v_x for which we have estimates that do not depend on the actual grid. Thus we may provide a global estimate for w_x which does not depend on ν.

Consider now the initial-boundary value problem of

$$\begin{aligned}\omega_{\overline{t}} = (E_x a)\omega_{x\overline{x}} + \frac{1}{2}(E_x b)\omega_x + \left(\frac{1}{2} E_x b + a_x\right)\omega_{\overline{x}} \\ + \left(E_x c + \frac{1}{2} b_x\right)\omega + \frac{1}{2} b_x E_x^{-1} \omega\end{aligned}$$

on the internal nodes of R_2 and $\omega = v_x - w$ on the parabolic boundary nodes of R_2. By corollary 8.18 this problem has a unique solution and since $\left(E_x c + \frac{1}{2} b_x\right) + \frac{1}{2} b_x \leq 0$ we may estimate

$$|\omega| \leq \max_{\delta' R_2} |\omega|$$

on R_2. But $|\omega| \leq |w| + |v_x|$ on the parabolic boundary of R_2 and since we have a global estimate of w and an estimate of v within R_4 that are both independent of ν we can provide an estimate of ω on R_2 which is independent of ν. We can also estimate ω_x relative to ω on the slightly smaller rectangle of $R = R_1$. By corollary 8.18 $v_x = w + \omega$ since $w + \omega$ solves the equation

$$(w+\omega)_{\overline{t}} = (E_x a)(w+\omega)_{x\overline{x}} + \frac{1}{2}(E_x b)(w+\omega)_x + \left(\frac{1}{2}E_x b + a_x\right)(w+\omega)_{\overline{x}} + \left(E_x c + \frac{1}{2}b_x\right)(w+\omega) + \frac{1}{2}b_x E_x^{-1}(w+\omega) + c_x v$$

within R_2 and $w + \omega = v_x$ on the parabolic boundary nodes of R_2. Thus we can bound $v_{xx} = w_x + \omega_x$ on R_1 and the bound will be independent of ν.

In summary: Given a closed rectangle R we have the ability to estimate v, v_x and v_{xx} on R and the bound does not depend on ν. We can repeat this procedure indefinitely or more precisely as long as the sum of the current set of creation coefficients is negative $\tilde{c}_1 + \tilde{c}_2 < 0$ and our relative estimate applies. But this is not a severe restriction. Let $\tilde{c}_1^j, \tilde{c}_2^j$ be the creation coefficients of the equation for $\delta_x^j v$. Then $\tilde{c}_1^j + \tilde{c}_2^j$ has the following structure

$$\tilde{c}_1^j + \tilde{c}_2^j = E_x^j c + \text{sum of various divided differences of } a,\ b, \text{ and } c$$

as can be seen by studying the relation between $\tilde{c}_i^{j+1}$ and $\tilde{c}^j$; and as noted in the beginning of this chapter we are free to adjust the value of c downward !

Thus given $m = 2n$ for some integer value of n we may derive the equations for $\delta_x^j v$ and study the demands placed on all the creation coefficients, return to the differential equation and adjust c downwards so that $\sup(\tilde{c}_1^j + \tilde{c}_2^j) < 0$. Then we follow the previous arguments and obtain for each point of Ω_ν an estimate of $\delta_x^j v$ that is independent of ν. The various time derivatives of v are estimated by applying the triangle inequality to the equations of $\delta_x^j v$. Observe how we immediately have a bound for $v_{\overline{t}}$ since

$$|v_{\overline{t}}| = \left| a v_{x\overline{x}} + \frac{1}{2} b v_x + \frac{1}{2} b v_{\overline{x}} + c v \right|.$$

Similarly we exploit the fact that $v_{\overline{t}x} = (v_x)_{\overline{t}}$ can be expressed in terms of v_{xxx}, v_{xx}, v_x, and v to bound $v_{\overline{t}x}$ on every compact set independently of ν. In general $\delta_t^i \delta_x^j v$ can be estimated in terms of $\delta_x^r v, r = 0, 1, 2, \ldots 2i + j$.

In summary: Given p it is possible to adjust c so that it is possible to bound all divided differences of order less than or equal to $p + 1$ independently of the grid and then apply the machinery of chapter 2 and generate a function $u \in C^p(\Omega \cup \delta''\Omega)$ such that

$$u_t = a u_{xx} + b u_x + c u + f$$

within $\Omega \cup \delta''\Omega$. As usual u is given by

$$u(x,t) = \lim_{\nu \to \infty} v^\nu(x,t)$$

at all points in $\Sigma \cap (\Omega \cup \delta''\Omega)$ and then extended continuously to the entire set of $\Omega \cup \delta''\Omega$.

We claim that $u : \Omega \cup \delta''\Omega \to \mathbb{R}$ is consistent with the initial-boundary condition in the sense that $u(x,t) \to g(p)$ for $(x,t) \to p$ with $(x,t) \in \Omega \cup \delta''\Omega$ and $p \in \delta'\Omega$.

The proof is virtually identical to proof of the similar statement for the special case of the heat equation. We still need to treat three or rather two separate cases: the initial line, the left hand side and similarly the right hand side of the parabolic boundary. Basically these proofs hinged on our ability to find barrier functions w such that $w > 0$ and $L(w) = w_x x - w_{\overline{t}} < 0$. We studied auxiliary functions of the type

$$\phi = g(p) - \epsilon - Cw - v^{\nu}$$

and showed that $\phi \leq 0$. Since $L(\phi) = \phi_{x\overline{x}} - \phi_{\overline{t}} = -CL(w) > 0$ the maximum of ϕ could not be achieved at an internal node and we only had to verify $\phi \leq 0$ at parabolic boundary nodes.

In our more general case of $L(w) = aw_{x\overline{x}} + bw_x + cw - w_{\overline{t}}$ with $c < 0$ we can still apply the same technique. If we can find an auxiliary function w such that $L(w) < 0$ then $L(\phi) > 0$ and any internal maximum of ϕ will be non-positive and thus in order to show $\phi \leq 0$ it still suffices to verify $\phi \leq 0$ at the parabolic boundary nodes. This is exactly what we did in the case of the heat equation.

Let us be a bit more precise:

8.28 Theorem. *If $p = (x_0, 0)$ lies on the initial line of Ω then*

$$u(x,t) \to g(p)$$

for $(x,t) \to p$ with $(x,t) \in \Omega$.

Proof. Let $w(x,t) = (x - x_0)^2 + Qt$ with $Q > 0$ to be determined shortly. Then

$$w_x = 2(x - x_0) + h \qquad w_{\overline{x}} = 2(x - x_0) - h$$
$$w_{x\overline{x}} = 2 \qquad w_{\overline{t}} = Q$$

and therefore

$$\begin{aligned} L(w) = aw_{x\overline{x}} + \frac{1}{2}bw_x + \frac{1}{2}bw_{\overline{x}} + cw - w_{\overline{t}} &= 2a + 2b(x - x_0) + cw - w_{\overline{t}} \\ &\leq 2\sup a + 2\|b\|_\infty \max_{x \in \overline{\Omega}} |x - x_0| - Q \end{aligned}$$

since $c < 0$. Obviously we can choose $Q > 0$ such that $L(w) < 0$ and now the proof proceeds exactly as in the case of the heat equation, lemma 7.21.

Similarly

8.29 Theorem. *If $p = (\phi_i(t_1), t_1)$ with $t_1 > 0$ has a barrier function then*

$$u(x,t) \to g(p)$$

for $(x,t) \to p$ with $(x,t) \in \Omega \cup \delta''\Omega$ and $t \leq t_1$.

Proof. The proof is completely identical to the proof of 7.24. The real problem lies in the construction of a barrier function which we shall address shortly.

8.30 Theorem. *If $p = (\phi_i(t_1), t_1)$ with $t_1 > 0$ then*

$$u(x,t) \to g(p)$$

for $(x,t) \to p$ with $(x,t) \in \Omega$ and $t \geq t_1$.

Proof. Once again the proof is quite similar to the proof of lemma 7.26. We merely need to adjust the value of $Q > 0$ such that the function

$$w(x,t) = (x - x_1)^2 + Q(t - t_1)$$

has $L(w) < 0$. Then we proceed in exactly the same fashion.

Petrowski [2] has given barrier functions for the heat equation and $\phi_i, i = 1, 2$ Lipschitz continuous. His barrier-functions are smooth and satisfy the differential inequality

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} < 0.$$

We claim that a barrier function for the heat equation can easily be transformed into a barrier function for our general equation. The key is the following lemma.

8.31 Lemma. *Let a, b, and c be real valued constants with $a > 0$. If $\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} < 0$ then it is possible to choose $p, q \in \mathbb{R}$ such that*

$$a\frac{\partial^2 v}{\partial x^2} + b\frac{\partial v}{\partial x} + cv - \frac{\partial v}{\partial t} < 0$$

where $v(x,t) = u\left(a^{-\frac{1}{2}}x, t\right) e^{px+qt}$.

Proof. Let $w(x,t) = u\left(a^{-\frac{1}{2}}x, t\right)$ then

$$a\frac{\partial^2 w}{\partial x^2}(x,t) - \frac{\partial w}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}\left(a^{-\frac{1}{2}}x, t\right) - \frac{\partial u}{\partial t}\left(a^{-\frac{1}{2}}x, t\right) < 0.$$

Let $v(x,t) = w(x,t)e^{px+qt}$ then

$$\frac{\partial v}{\partial x} = \left(\frac{\partial w}{\partial x} + pw\right) e^{px+qt} \qquad \frac{\partial v}{\partial t} = \left(\frac{\partial w}{\partial t} + pw\right) e^{px+qt}$$

and

$$\frac{\partial^2 v}{\partial x^2} = \left(\frac{\partial^2 w}{\partial x^2} + 2p\frac{\partial w}{\partial x} + p^2 w\right) e^{px+qt}$$

so that

$$a\frac{\partial^2 v}{\partial x^2} + b\frac{\partial v}{\partial x} + cv$$
$$= \left(a\frac{\partial^2 w}{\partial x^2} - \frac{\partial w}{\partial t} + (2pa+b)w_x + (ap^2+bp+c-q)w \right) e^{px+qt}.$$

Now we choose $p = \frac{-b}{2a}$ so that $2pa + b = 0$ and set

$$q = ap^2 + bp + c = \frac{b^2}{4a} - \frac{b^2}{2a} + c = -\frac{b^2 - 4ac}{4a}.$$

Then $ap^2 + bp + c - q = 0$ and

$$a\frac{\partial^2 v}{\partial x^2} + b\frac{\partial v}{\partial x} + cv = aw_{xx} - w_t < 0$$

which completes the proof of this lemma.

Please note that

$$v(x,t) > 0 \Leftrightarrow u\left(a^{-\frac{1}{2}x,t}, t\right) > 0$$

and similarly

$$v(x,t) = 0 \Leftrightarrow u\left(a^{-\frac{1}{2}x,t}, t\right) = 0.$$

8.32 Lemma. *If*

$$a(x_0,t_0)\frac{\partial^2 u}{\partial x^2}(x,t) + b(x_0,t_0)\frac{\partial u}{\partial x}(x,t) + c(x_0,t_0)u(x,t) - \frac{\partial u}{\partial t}(x,t) < 0$$

in some neighbourhood U of (x_0,t_0) then

$$a(x,t)\frac{\partial^2 u}{\partial x^2}(x,t) + b(x,t)\frac{\partial u}{\partial x}(x,t) + c(x,t)u(x,t) - \frac{\partial u}{\partial t}(x,t) < 0$$

in some smaller neighbourhood V of (x_0,t_0).

Proof. The proof relies on the continuity of the coefficients a, b, and c. We may assume that U is compact, so that the continuous functions $\frac{\partial^2 u}{\partial x^2}, \frac{\partial u}{\partial x}, u$ are bounded on U by M. By continuity there exists $\epsilon > 0$ such that

$$a(x_0,t_0)\frac{\partial^2 u}{\partial x^2}(x,t) + b(x_0,t_0)\frac{\partial u}{\partial x}(x,t) + c(x_0,t_0)u(x,t) - \frac{\partial u}{\partial t}(x,t) < -\epsilon$$

on U. By the uniform continuity of a, b, and c on U we can find a neighbourhood V of (x_0,t_0) such that

$$|a(x,t) - a(x_0,t_0)| < \frac{\epsilon}{6M}$$

and similarly for b and c. Thus

$$
\begin{aligned}
&-\frac{\epsilon}{2} < a(x,t)\frac{\partial^2 u}{\partial x^2}(x,t) + b(x,t)\frac{\partial u}{\partial x}(x,t) + c(x,t)u(x,t) - \frac{\partial u}{\partial t}(x,t) \\
&-\left(a(x_0,t_0)\frac{\partial^2 u}{\partial x^2}(x,t) + b(x_0,t_0)\frac{\partial u}{\partial x}(x,t) + c(x_0,t_0)u(x,t) - \frac{\partial u}{\partial t}(x,t)\right) < \frac{\epsilon}{2}
\end{aligned}
$$

within V and we are done.

We conclude this chapter by stating the main result explicitly

8.33 Theorem. *If a barrier function can be found for each point of the left hand side and for each point of the right hand side of the parabolic boundary of Ω then the initial-boundary value problem of*

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + cu + f$$

within $\Omega \cup \delta''\Omega$ and

$$u(x,t) \to g(p) \quad \text{as} \quad (x,t) \to p \quad \text{with} \quad (x,t) \in \Omega \cup \delta''\Omega$$

at each point of parabolic boundary has a unique solution

$$u \in C^\infty(\Omega \cup \delta''\Omega) \cap C(\overline{\Omega}).$$

Proof. We have already constructed a solution $u \in C^\infty(\Omega \cup \delta''\Omega)$ such that

$$u(x,t) \to g(p) \quad \text{as} \quad (x,t) \to p \quad \text{with} \quad (x,t) \in \Omega \cup \delta''\Omega$$

at each point of the parabolic boundary. Thus $u \in C^\infty(\Omega \cup \delta''\Omega) \cap C(\overline{\Omega})$ and the estimate of theorem 5.6 implies uniqueness.

Chapter 9

Conclusion

We have solved the pure initial value problem as well as the initial-boundary value problem for the differential equation

$$u_t - au_{xx} - bu_x - cu = f$$

on a fairly general type of bounded domain. We have rediscovered a set of existence and uniqueness theorems for these problems. We have derived maximum principles for the finite difference equations and demonstrated how to use them to determine the rate of convergence. We have used smooth maximum principles to show that the solutions depend continuously on the initial-boundary conditions as well as on the coefficients of the equation. We also derived a Taylor-like expansion of the error for the heat equation.

It is possible to learn a great deal about differential equations by studying finite difference methods.

Treating the solution of the differential equation as a limit of solutions to a sequence of difference equations allows us to gain information about the limit by studying the individual elements. As an example I would like to emphasize that the smooth maximum principle of lemma 3.12 was inherited directly from the corresponding discrete maximum principle of theorem 3.8. Maximum principles are fairly easy to derive for difference equations because we can actually compute the solution or at the very least write down a solution formula. This is much more difficult to do for the differential equation itself, especially if the domain has a non-trivial boundary.

On a bounded domain, i.e. with a *finite* number of linear equations, a discrete maximum principle can be used to establish the existence of solutions to the difference equation in question. This is more difficult for unbounded domains and implicit equations because we need to invert a bounded operator on an infinite dimensional Banach-space.

Not every finite difference method is suitable for the study of the corresponding differential equation. We had to abandon the explicit method while studying the initial-boundary value problem because $L(v) = v_{x\overline{x}} - v_t = 0$ did not imply that $L(v^2) \geq 0$. This useful property is shared by the implicit method and the solutions of the heat equation.

I hope to be able to apply and develop this simple technique to more complicated equations than those studied here. I would like to develop the idea of an invariant scheme further. The differential equations that admit an invariant scheme are those for which it is at least theoretically possible to provide an elementary solution of the Cauchy problem. It is merely a question of identifying the invariant form and discovering a maximum principle for this form as we did in chapters 3 and 8. I realize that in general the problems will be formidable; in particular for non-linear problems where we shall not be able to do away with inhomogeneous terms as we did repeatedly in chapter 8. But still I feel that it is quite satisfying to be able to provide a simple proof of nontrivial existence theorems.

I would like to study the behaviour of the solutions of our difference equation at the parabolic boundary. The maximum principles of this thesis are clearly not the best possible since they do not take into account the distance between the grid point and the parabolic boundary. I hope that such a study would yield information about the behaviour of the derivatives of the solutions to the corresponding differential equation at the boundary. It would be nice to be able to derive a priori estimates of the various derivatives. This would allow the error analysis of chapter 4 to be carried out for more complicated initial-boundary value problems.

Bibliography

[1] John, Fritz, *Partial Differential Equations*,
Springer-Verlag, New York, 1982, pp. 227-233.

[2] Petrowski, I.G, *Partial Differential Equations*,
W.B. Saunders Company, Philadelphia, 1967, pp. 354-360 and 363-368.

[3] Østerby, Ole, *Numerical Analysis and Parabolic Equations*,
Department of Computer Science, University of Aarhus, 2003, pp. 43-66.