A Lative Logic View of the Filioque Addition

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Augustinus:
If I fail, I am.
Si enim fallor, sum. (De civitate Dei)

Descartes:
I think. Therefore I am.
Cogito, ergo sum.

I know - I am in possession of knowledge
I am - I am capable of knowing

If I know, I am.
If I am, I do not necessarily know.

When I think, I reveal knowledge to myself, and I may want to modify it, from time to time.

A husband is married to his wife.
A wife is married to her husband.

"x `is_married_to` y" means legally that "y `is_married_to` x"

This is an example in society where we are moving away from gender subordination, so we have a non-commutativity that over time moves towards being commutative.

There are also non-commutative subordinations that are sometimes treated as being commutative.

"`from` x `and_from` y" and "`from` y `and_from` x"
are not necessarily the same.

A syntactic expression like "`from` (x `and` y)" could be treated as the same as "`from` (y `and` x)" if "`and`" is understood as commutative.

"`from`" in "`from(...)`" is blind for "`...`".

An expression like "`from` x `and` y" is tricky if we do not recognize the parenthesis.

unless, lat. nisi

$Q(x, y)$: knowing x unless [knowing] y

Q is not eo ipso commutative, so if in some context we want both to hold, i.e., $Q(x, y)$ and $Q(y, x)$ at the same time, we need to do add both explicitly.

no man knoweth the Son, but the Father; neither knoweth any man the Father, save the Son
nemo novit Filium nisi Pater neque Patrem quis novit nisi Filius
(St. Matthew 11:27)

"some S is P" is commutative if written in first-order logic as $\exists x.Sx\&Px$, but is this rewriting really appropriate, and is first-order logic indeed too poor as a language?

The distribution of negation $\neg$ over existence $\exists$ is also very doubtful.

- A negation operator $\neg$ can be applied to the term $P(x)$, which indeed is constructed by the operator P, so that $\neg P(x)$ and $P(x)$ are of the same sort, as terms.
- However, as $\exists x.P(x)$ is not a term, but is expected to be a sentence, and it is very questionable whether $\neg$ in $\neg\exists x.P(x)$ and $\exists x.\neg P(x)$ really is the same symbol.
- In $\exists x.\neg P(x)$, it acts an operator, changing a term to term, but in $\neg\exists x.P(x)$ it changes a sentence to a sentence, so it is strictly speaking not an 'operator'.
- Variables may be substituted by terms, but 'sentential' variables make no sense with respect to substitution.

Having no typing and no formal distinction between terms and sentences allows for sentence constructions that implicitly mixes sorts.

Perrone's (1995) "collection of axiomatizations" is an indexing, not using an index set of sorts, but as a way of indexing logics.

Perrone creates a sentence (in an equational style logic) like $S_{g_i}(x_{g_i} + y_{g_i}) = S_{g_j}(x_{g_j} + y_{g_j})$, and then takes the terms $S_{g_i}(x_{g_i} + y_{g_i})$ and $S_{g_j}(x_{g_j} + y_{g_j})$ from different logics, creating a sentences in a common logic for which their is not necessarily a counterpart in the "collection of axiomatizations".

Unsorted "fons et origo" first-order logic and axiomatic set theory easily allows for "mixed bags" in particular when dealing with terms and sentences, but also when mixing truth and provability.

Church's (1940) distinction between the "sort of the sorts of terms" from the "sort of sentences", was implicitly observed by Schönfinkel (1924) in his unsorted approach, but has not matured in modern type theory (not even in Homotopy Type Theory).

Using notations from Kleene's "Metamathematics", a predicate symbol A and a predicate $A(x)$ invites to speak about "$A(x)$ is provable" and using the notation "$\vdash A(x)$".

However, proceeding to create a "metamathematical proposition" $\mathfrak{R}(x, Y)$, representing "Y is a proof of $A(x)$", then allowing to write

$$(\exists Y)\mathfrak{R}(x, Y) \;\equiv\, \vdash A(x)$$

and at the same time wondering "What is the nature of the predicate $\mathfrak{R}(x, Y)$?", requires a by-passing by saying it must be an "effectively decidable" metamathematical predicate, and that "there must be a decision procedure or algorithm for the question whether $\mathfrak{R}(x, Y)$ holds".

Mathematical propositions and metamathematical propositions are thus allowed to be in the same bag, and in Gödel's work there is frequent use of that degree of freedom to mix bags.

In fact, Gödel's "incompleteness" should not be seen as a theorem. It's a paradox.

Aristotle does not clearly distinguish between truth and provability.

In his Prior Analytics, Aristotle says "a true conclusion may come through what is false". What is here a "true conclusion"?

In propositional logic, if B is true then $False \Rightarrow B$ is also true. Is B the conclusion, or is "$False \Rightarrow B$ is true" the conclusion, or is it in fact "$\vdash False \Rightarrow B$ is true", i.e., "$False \Rightarrow B$ is provable"?

Aristotle also speaks about "the same terms", and then the question is what he means by a "term". Saying "positive terms in positive syllogisms" indicates that terms are sentences, but the two "positive" have different meanings.

In his statement "it is impossible that the same thing should be necessitated by the being and by the not-being of the same thing", Aristotle then mixes truth and provability, and trying to make that into a "sentence".

Aristotle's final statement "just as if it were proved through three terms" also clearly reveals how Aristotle becomes intertwined since he does not separate truth from provability.

In natural language we mix these things all the time.

We have used and we still use (natural and native) language to speak and write about the Word.

However, we should not abuse language to speak and write about the Word.

Can Language 'explain' the Word, or are writings written in Language just written representations of the Word?

Is there a “correct and complete” way to explain and/or write?

There are canonical writings,
but is there a canonical way to write about these writings?

There is perhaps an ecumenic way to write about the writings of the writings,
but not an ecumenic way to write about the writings?

This changes over times, as later ecumenic councils look backwards, affirming, or not affirming, what is and what isn't.

The ecumenic councils 869-870 and 879-880 were critical, and not just because of 'Filioque'.

Can Natural Language explain Logic?
Can Logic explain Natural Language?

"Language (structure) and Word", and "Language (structure) and Church", is that the same "Language (structure)"?
Is the related "Language and Logic" the same?

Maybe it is so that logic and reason can be enriched by Something, or Spirituque, that is in the Word and which proceeds through Natural and Native Language?

My personal view is 'Yes', and this is the fundamental reason for work underlying this presentation.

Gloria Patri, et Filio, et Spiritui Sancto

'et' is non-commutative (?),
even if "three is one".

"qui ex Patre Filioque procedit",
but not "qui ex Filio Patreque procedit"

Filio Patreque makes no sense?

So -que is a non-commutative (logical) connective.

In two-valued logic, Boolean algebras are the only options. Concerning three-valued logic, the additional truth value doesn't have to be "in between", but can be a '`not known`', like in WHO's classification of functioning (ICF).

Quantales as algebraic structures have turned out suitable in these respects, because terms functors over certain quantale related categories can be constructed, since these categories are monoidal bi-closed.

Non-commutative quantales can further be either left-sided or right-sided.

Obviously, Aristotle time mathematics was not at all aware of these things. Boole didn't realize it. Many-valued approaches during early 20th century were always commutative and lattice oriented. Quantales do not appear until during the second half of the 20th century.

The Arian controversy ... "procedit a solo Filio" ... Augustinus' "nec a solo Filio missum est, sed a Patre quoque" ... with culmination in the Toledo council 589, where the Spanish Church stood up against the Arianist Visigoths.

The Byzantine triadology still rejects any causative participation from the Son in the proceeding of the Spirit.

The Latin "procedit", e.g. as in St. John 15:26, comes from the Greek "ekporeuomenon", and as related with the Aramaic "npq".

Translations may be slightly different in respective languages, but that is indeed how it was done at that time. Clearly, efforts to translate sentences expressed in natural languages like Aramaic, Greek and Latin to corresponding sentences in logically enriched natural languages, must then respect both syntactic as well as semantic aspects.

The way we build and dissect clauses in native and natural language then has bearing also on the causality aspect of "procedit", and how logically to handle the "-que" as a connective.

For Augustinus, and the way he chose to formulate himself against "procedit a solo Filio", it was maybe just a matter of strategy?

The Spirit isn't 'given' until through the Son. (St. John 7:39)

‘Causality’ (relational) and ‘process’ (flow or sequence) should also not be confused. Causality is in logic better understood than process. The latter involves time.

Meaning and view of process may also change in translation.

As pointed out implicitly by Augustinus in his *De Sermone Domini In Monte* (394), Jerome did the modification of St. Matthew 6:6 from “and while you close the door” (Vetus Latina: claudentes ostia) to “and when thou hast shut thy door” (Vulgata: clauso ostio).

The reason for this change may be mostly unknown (?), but one may speculate that the reason for this change is liturgic, since a ceremony is always “sequential” in some sense, i.e., in the style of explaining “first do this, and then this, and then that”.

no one knows God unless He who knows manifests Himself
Deum nullus cognoscit, si non se indicat ipse qui novit
(Thomas Aquinas' Compendium, referring to Augustine's commentary on John)

God reveals will. Is that a state and/or in a moment?
State of will and moment of will may not be the same?

Propositional revelations are truths revealed by God but they are not verified using human reason.
(Thomas Aquinas)

to speak, lat. loqui
speech, lat. locutio

In interior locution, something is spoken, something is “delivered”, or something “is just there”, exists, or simply “is”.

Would we say “it is a dialogue”, or would we just say “it is”?
In divine revelation, “dialogue” is of different type than “dialogue” between humans?

Not understanding the distinction between being and existence, and neglecting the dialogic nature presentation, is the basic logical weakness of Areios’ and Sabellius’ existence “proofs”, and actually the weakness of Aristotle’s logic as a whole.

In the following, and as written in natural language, each statement can individually be observed and understood.

There is only one God.
The Father is God.
The Son is God.
The Father is not the Son.
The Holy Spirit is God.
The Holy Spirit is not the Father.
The Holy Spirit is not the Son.

However, if we allow them to appear as comglomerated statements, we have to very careful, since we are probably hiding much of the conglomeration.

Areios' tried something like the following:

There is only one God.
The Father is God.
The Father is not the Son.
Therefore the Son is not God.

Sabellius' tried something like the following:

There is only one God.
The Father is God.
The Son is God.
Therefore the Father is the Son.

... and the reasoning machinery makes things go wrong. Church says it's heresy, which is a decision up to the Church, but it is also "mathematical and logical heresy".

As conjecture before, Word has nothing to learn from the structure of Natural Language and Logic, but Natural Language and Logic may learn a few things from the structure of Word?

Or is Word just content and without structure?

What we [can] know [about God and goodness] is what has been revealed [to us]. (Romans 1:19)

quia quod notum est Dei manifestum est in illis Deus enim illis manifestavit (Vulgata)
Denn was man von Gott weiß, ist ihnen offenbar; denn Gott hat es ihnen offenbart, (Luther 1545)
Because that which may be known of God is manifest in them; for God hath shewed it unto them. (King James Version 1611)
Vad man kan känna om Gud är nämligen uppenbart bland dem; Gud har ju uppenbarat det för dem. (Svenska 1917)
sentähden että se, mikä Jumalasta voidaan tietää, on ilmeistä heidän keskuudessaan; sillä Jumala on sen heille ilmoittanut. (Raamattu 1933/38)

good/bad, [this] is good - [this] is bad
right/wrong, doing [things] right - doing [things] wrong
true/false, what is known [about something] is true - what is known [about something] is false

knowing what is good and knowing when doing good

Patrik's iron 7 from 130 meters didn't go into the green, but went out of bounds.

Patrik as a golfer is bad.
Patrik as a golfer did wrong.
"Patrik as a golfer is good." is false
"Patrik as a golfer is good." is wrong

It is true that I am wrong.
It is good to say that it is true that I am wrong.

Obviously, from logic point of view, all this make no sense at all, since we do not recognize types, and we do not make clear separation between term and sentence.

knowledge of [God's greatness and] goodness cannot come to humans except through the grace of divine revelation
cognitio [divinae magnitudinis] et bonitatis hominibus provenire non potest nisi per gratiam revelationis divinae
(Thomas Aquinas' Compendium, Chapter 8)

"doing good and knowing that" is good (or is it really?)
"doing good because of knowing it's good doing that" is bad
(Luther said something like "doing good is doing sin")

In social and health care for the ageing population, there is tendency to shift from co-morbidity to multi-morbidity.

There is thus not a "main disorder" to be treated "first", and then the other disorders are treated as dependent of the treatment of the "main disorder".

Nevertheless, disorders may related to cell, tissue, organ and organ system, so multi-morbidity will not imply commutativity.

Clinical guidelines and care recommendations are not logical, but rather based on numbers moved over from evidence-based medicine. Statistics is on population, and logic on individual. Health care hasn't solved this problem.

What we know about population is not what we know about individual.

Decisions about intervention and treatment are based mostly on disorder (WHO's ICD classification), but the effect of an intervention is typically measured with respect to maintenance or improvement of functioning (WHO's ICF classification).

Functioning proceeds from disorder?
Disorder proceeds from functioning?

Previous stroke, depression and hypertension treatment may indicate that a cognitive failure should be investigated also as a possible vascular dementia and not just a Alzheimer's disease. Inhibitor drugs have no effect for vascular dementia patients, just side-effects.

The "and" in assessment scales is usually modelled using incrementing numbers, so it becomes arithmetics rather than logics.

Logic is not spoken.
English is not spoken.
English is not read or written.
English is a language we use when we speak, and we when read and write.
English is the language we use when we speak about something, or say something, or write about something.
Logic is the language we use when we "speak" (formulate) sentences and statements.

verbal and non-verbal

Is "verbal" something that can be written in natural language, and "non-verbal" something than cannot?
Does "verbal" adhere to a grammar?
The "grammar" of logic and natural language is different.
Is there a "universal grammar" embracing both logic and natural language?
Can there be a "universal grammar" embracing both logic and natural language?
Is it desirable to have a "universal grammar" embracing both logic and natural language?

How do we read/write/speak [about] the Word?
How do we read/write/speak [being] in Church?

'Lative' is "motion", motion 'to' and 'from', so when terms appear in sentences, terms 'move into' sentence, and sentences 'move away from' terms. In comparison, 'ablative' is "motion away", and nominative is static. The lative locative case (casus) indeed represents "motion", whereas e.g. a vocative case is identification with address.

- "Lative logic" is more about "lativity" between various components and building blocks of a logic as a categorical object, rather than traditionally creating "yet another logic".
- It is also distinct from the "fons et origo" foundational logic, where the roles of metalanguage and object language may be blurred.
- This approach to logic assumes category theory as its metalanguage, and leans on having signatures as a pillar and starting point for "terms", which in turn are needed in "sentences", and so on.

- Adapting a strictly categorical framework, as a chosen metalanguage, enables us to be very precise about the distinction between terms and sentences, where 'boolean' operator symbols, i.e. where the codomain sort of the operator is a 'boolean' sort, become part of the underlying signature.
- Implication is not introduced as an operator in the signature, nor as a short name using existing operators, but will appear as integrated into our sentence functors.

- We may produce a sentence as a pair $(P(x), Q(y))$ of terms, where they are produced by its own term functors.
- Intuitively, this corresponds to "$P(x)$ is inferred by $Q(y)$".
- The 'pairing operation', i.e., the 'implication', is not given in the underlying signature as an operator, but appears as the result of functor composition and product within a 'sentence constructor'.

Signatures

- The previous talk was using a strictly mathematical, and a 'monoidal biclosed categorical' notation for signatures. Here we adopt the more 'computationally intuitive' notation of a signature, but the content and concept is the same as for the strict one.
- A many-sorted signature $\Sigma = (S, \Omega)$ consists of a set S of sorts (or types), and a tupled set $\Omega = (\Omega_{\mathtt{s}})_{\mathtt{s} \in S}$ of operators. Operators in $\Omega_{\mathtt{s}}$ are written as $\omega : \mathtt{s}_1 \times \cdots \times \mathtt{s}_n \to \mathtt{s}$.

Signatures over underlying categories

- We indeed restrict to quantales $\mathfrak{Q}$ that are commutative and unital, as this makes the Goguen category $\mathtt{Set}(\mathfrak{Q})$ to be a symmetric monoidal closed category and therefore also biclosed.
- This Goguen category carries all structure needed for modelling uncertainty using underlying categories for fuzzy terms over appropriate signatures.
- A signature $(S, (\Omega, \alpha))$ over $\mathtt{Set}(\mathfrak{Q})$ then typically has S as a crisp set, and $\alpha : \Omega \to Q$ then assigns uncertain values to operators.

Highlights of the term construction

We use the notation

$$\Omega^{\mathtt{s}_1 \times \cdots \times \mathtt{s}_n \to \mathtt{s}}$$

for the set of operators $\omega : \mathtt{s}_1 \times \cdots \times \mathtt{s}_n \to \mathtt{s}$ (in $\Omega_{\mathtt{s}}$) and

$$\Omega^{\to \mathtt{s}}$$

for the set of constants $\omega : \to \mathtt{s}$ (also in $\Omega_{\mathtt{s}}$), so that we may write

$$\Omega_{\mathtt{s}} = \coprod_{\substack{\mathtt{s}_1, \ldots, \mathtt{s}_n \\ n \leq k}} \Omega^{\mathtt{s}_1 \times \cdots \times \mathtt{s}_n \to \mathtt{s}}.$$

For the term functor construction over $\mathtt{Set}(\mathfrak{Q})$ we need objects

$$(\Omega^{\mathtt{s}_1 \times \cdots \times \mathtt{s}_n \to \mathtt{s}}, \alpha^{\mathtt{s}_1 \times \cdots \times \mathtt{s}_n \to \mathtt{s}})$$

for the operators $\omega : \mathtt{s}_1 \times \cdots \times \mathtt{s}_n \to \mathtt{s}$, and

$$(\Omega^{\to \mathtt{s}}, \alpha^{\to \mathtt{s}})$$

for the constants $\omega :\to \mathtt{s}$.

The term functor construction over Set

$$\Psi_{\mathtt{m},\mathtt{s}}((X_{\mathtt{t}})_{\mathtt{t}\in S}) = \Omega^{\mathtt{s}_1\times\ldots\times\mathtt{s}_n\to\mathtt{s}} \otimes \bigotimes_{i=1,\ldots,n} X_{\mathtt{s}_i},$$

changes over Set($\mathfrak{Q}$) to

$$\Psi_{\mathtt{m},\mathtt{s}}(((X_{\mathtt{t}},\delta_{\mathtt{t}}))_{\mathtt{t}\in S}) = (\Omega^{\mathtt{s}_1\times\ldots\times\mathtt{s}_n\to\mathtt{s}},\ \alpha^{\mathtt{s}_1\times\ldots\times\mathtt{s}_n\to\mathtt{s}}) \otimes \bigotimes_{i=1,\ldots,n} (X_{\mathtt{s}_i},\delta_{\mathtt{s}_i})$$

$$= (\Omega^{\mathtt{s}_1\times\ldots\times\mathtt{s}_n\to\mathtt{s}} \times \prod_{i=1,\ldots,n} X_{\mathtt{s}_i},\ \alpha^{\mathtt{s}_1\times\ldots\times\mathtt{s}_n\to\mathtt{s}} \odot \bigodot_{i=1,\ldots,n} \delta_{\mathtt{s}_i}).$$

The inductive steps in the construction:

- $\mathsf{T}^1_{\Sigma,\mathtt{s}} = \coprod_{\mathtt{m}\in\hat{\mathtt{S}}} \Psi_{\mathtt{m},\mathtt{s}}$
- $\mathsf{T}^{\iota}_{\Sigma,\mathtt{s}} X_{\mathtt{S}} = \coprod_{\mathtt{m}\in\hat{\mathtt{S}}} \Psi_{\mathtt{m},\mathtt{s}}(\mathsf{T}^{\iota-1}_{\Sigma,\mathtt{t}} X_{\mathtt{S}} \sqcup X_{\mathtt{t}})_{\mathtt{t}\in\mathtt{S}})$, for $\iota > 1$

We have $\mathsf{T}^{\iota}_{\Sigma} X_{\mathtt{S}} = (\mathsf{T}^{\iota}_{\Sigma,\mathtt{s}} X_{\mathtt{S}})_{\mathtt{s}\in\mathtt{S}}$. Further, $(\mathsf{T}^{\iota}_{\Sigma})_{\iota>0}$ is an inductive system of endofunctors, and the inductive limit $\mathsf{F} = \mathrm{ind}\varinjlim \mathsf{T}^{\iota}_{\Sigma}$ exists.

The final term functor:

- $\mathsf{T}_{\Sigma} = \mathsf{F} \sqcup \mathsf{id}_{\mathtt{Set}_{\mathtt{S}}}$

We also have $\mathsf{T}_{\Sigma} X_{\mathtt{S}} = (\mathsf{T}_{\Sigma,\mathtt{s}} X_{\mathtt{S}})_{\mathtt{s}\in\mathtt{S}}$.

Terms and ground terms

In order to proceed towards creating sentences, we need the so called 'ground terms' produced by the term monad.

- $\Sigma_0 = (S_0, \Omega_0)$ over $\mathtt{Set}$
- $\mathbf{T}_{\Sigma_0}$ term monad over $\mathtt{Set}_{S_0}$
- $\mathsf{T}_{\Sigma_0}\emptyset_{S_0}$ is the set of 'ground terms'

‘Predicate’ symbols as operators in a signature

- We now proceed to clearly separate views of terms and sentences, respectively, in propositional logic and predicate logic.
- In order to introduce ‘predicate’ symbols as operators in a specific signature, we assume that Σ contains a sort `bool`, which does not appear in connection with any operator in Ω_0, i.e., we set $S = S_0 \cup \{\mathtt{bool}\}$, $\mathtt{bool} \notin S_0$, and $\Omega = \Omega_0$.
- This means that $\mathsf{T}_{\Sigma,\mathtt{bool}} X_S = X_{\mathtt{bool}}$, and for any substitution $\sigma_S : X_S \to \mathsf{T}_\Sigma X_S$, we have $\sigma_{\mathtt{bool}}(x) = x$ for all $x \in X_{\mathtt{bool}}$.
- `bool` is kind of the “predicates as terms” sort.

Sentences in propositional logic

Signature:

- Let $\Sigma_{PL} = (S_{PL}, \Omega_{PL})$, where $S_{PL} = S$ and $\Omega_{PL} = \{\mathtt{F}, \mathtt{T} :\to \mathtt{bool}, \& : \mathtt{bool} \times \mathtt{bool} \to \mathtt{bool}, \neg : \mathtt{bool} \to \mathtt{bool}\} \cup \{\mathtt{P}_i : \mathtt{s}_{i_1} \times \cdots \times \mathtt{s}_{i_n} \to \mathtt{bool} \mid i \in I, \mathtt{s}_{i_j} \in S\}$.
- Similarly as `bool` leading to no additional terms, except for additional variables being terms when using Σ, the sorts in S_{PL}, other than `bool`, will lead to no additional terms except variables.
- Adding 'predicates' as operators even if they produce no terms seems superfluous at first sight, but the justification is seen when we compose these term functors with T_Σ.

- For the sentence functor, we need the ‘tuple selecting’ functor $\mathrm{arg}^{\mathtt{s}}\colon \mathtt{C}_S \to \mathtt{C}$ such that $\mathrm{arg}^{\mathtt{s}} X_S = X_{\mathtt{s}}$ and $\mathrm{arg}^{\mathtt{s}} f_S = f_{\mathtt{s}}$.
- We also need the ‘variables ignoring’ functor $\phi^{\mathtt{s}} : \mathtt{Set}_S \to \mathtt{Set}_S$ such that $\phi^{\mathtt{s}} X_S = X'_S$, where for all $\mathtt{t} \in S \backslash \{\mathtt{s}\}$ we have $X'_{\mathtt{t}} = \emptyset$, and $X'_{\mathtt{s}} = X_{\mathtt{s}}$. Actions on morphisms are defined in the obvious way.

Propositional logic ‘formulas’ as sentences:

- $\mathsf{Sen}_{PL} = \mathrm{arg}^{\mathtt{bool}} \circ \mathsf{T}_{\Sigma_{PL}} \circ \phi^{\mathtt{bool}}$

Notational flexibility and selectivity ...

- $\Sigma_{PL\setminus\neg}$ is the signature where the operator $\neg$ is removed, and $\Sigma_{PL\setminus\neg,\&}$ where both $\neg$ and & are removed
- $\bigcup_{\mathtt{s}\in S}(\mathsf{T}_{\Sigma,\mathtt{s}} \circ \phi^{S\setminus\mathtt{bool}})\emptyset_S$ is the set of all 'non-boolean' sorted terms, i.e., the "unsorted set" of all "ground terms", and corresponds to the so called the "Herbrand universe"
- $\bigcup_{\mathtt{s}\in S}(\mathsf{T}_{\Sigma,\mathtt{s}} \circ \phi^{S\setminus\mathtt{bool}})X_S$ is syntactically the set of all 'non-boolean' sorted terms, i.e., the "unsorted set" of all terms, and corresponds semantically to the "Herbrand interpretation"
- note also how $(\mathsf{arg}^{\mathtt{bool}} \circ \mathsf{T}_{\Sigma_{PL\setminus\neg,\&}} \circ \phi^{\mathtt{bool}})X_S = \{\mathtt{F}, \mathtt{T}\}$

The sentence functor for Horn clause logic (HCL)

$$\begin{aligned}\mathsf{Sen}_{HCL} &= (\mathsf{arg}^{\texttt{bool}})^2 \circ (((\mathsf{T}_{\Sigma_{PL\setminus\neg,\&}} \circ \mathsf{T}_\Sigma) \times (\mathsf{T}_{\Sigma_{PL\setminus\neg}} \circ \mathsf{T}_\Sigma)) \circ \phi^{S\setminus\texttt{bool}}) \\ &= (\mathsf{arg}^{\texttt{bool}})^2 \circ ((\mathsf{T}_{\Sigma_{PL\setminus\neg,\&}} \times \mathsf{T}_{\Sigma_{PL\setminus\neg}}) \circ \mathsf{T}_\Sigma \circ \phi^{S\setminus\texttt{bool}})\end{aligned}$$

- the pair $(h, b) \in \mathsf{Sen}_{HCL} X_S$, as a sentence representing the ‘Horn clause’, means that h is an ‘atom’ and b is a conjunction of ‘atoms’
- $(h, \texttt{T})$ is a ‘fact’
- $(\texttt{F}, b)$ is a ‘goal clause’
- $(\texttt{F}, \texttt{T})$ is a ‘failure’

Modus Ponens as an inference rule then looks more like ...

$$\frac{(\mathtt{F}, b) \quad (h, b)}{(h, \mathtt{T})}$$

This is correctly written since we use sentences only, i.e., not mixing terms and sentences in proof rules, but it is still informal since an inference rule involves 'theoremata'.

Anticipating the notion of 'theoremata' as a structured set of sentences, the following proof rule involves 'one-sentence theoremata' in the special case of having the theoremata functor being the powerset functor composed with the sentence functor.

$$\frac{\{(\mathtt{F}, b)\} \ddagger \{(h, b)\}}{\{(h, \mathtt{T})\}}$$

Variable substitutions within sentences

- $\sigma_S : \phi^{S\backslash\texttt{bool}} X_S \to \mathsf{T}_\Sigma \phi^{S\backslash\texttt{bool}} Y_S$
- $\mu \circ \mathsf{T}_\Sigma \sigma_S : \mathsf{T}_\Sigma \phi^{S\backslash\texttt{bool}} X_S \to \mathsf{T}_\Sigma \phi^{S\backslash\texttt{bool}} Y_S$

$$\sigma_S^{head} = \mathsf{T}_{\Sigma_{PL\backslash\neg,\&}}(\mu \circ \mathsf{T}_\Sigma \sigma_S) : (\mathsf{T}_{\Sigma_{PL\backslash\neg,\&}} \circ \mathsf{T}_\Sigma)\phi^{S\backslash\texttt{bool}} X_S \\ \to (\mathsf{T}_{\Sigma_{PL\backslash\neg,\&}} \circ \mathsf{T}_\Sigma)\phi^{S\backslash\texttt{bool}} Y_S$$

$$\sigma_S^{body} = \mathsf{T}_{\Sigma_{PL\backslash\neg}}(\mu \circ \mathsf{T}_\Sigma \sigma_S) : (\mathsf{T}_{\Sigma_{PL\backslash\neg}} \circ \mathsf{T}_\Sigma)\phi^{S\backslash\texttt{bool}} X_S \\ \to (\mathsf{T}_{\Sigma_{PL\backslash\neg}} \circ \mathsf{T}_\Sigma)\phi^{S\backslash\texttt{bool}} Y_S$$

$$(\sigma_S^{head}, \sigma_S^{body}) = (\mathsf{T}_{\Sigma_{PL\backslash\neg,\&}} \times \mathsf{T}_{\Sigma_{PL\backslash\neg}})(\mu \circ \mathsf{T}_\Sigma \sigma_S) : ((\mathsf{T}_{\Sigma_{PL\backslash\neg,\&}} \times \mathsf{T}_{\Sigma_{PL\backslash\neg}}) \circ \mathsf{T}_\Sigma)\phi^{S\backslash\mathtt{bool}} X_S \to ((\mathsf{T}_{\Sigma_{PL\backslash\neg,\&}} \times \mathsf{T}_{\Sigma_{PL\backslash\neg}}) \circ \mathsf{T}_\Sigma)\phi^{S\backslash\mathtt{bool}} Y_S$$

$$\sigma^{HC} = (\sigma_{\mathtt{bool}}^{head}, \sigma_{\mathtt{bool}}^{body}) : \mathsf{Sen}_{HCL} X_S \to \mathsf{Sen}_{HCL} Y_S$$

Lative Logic as an extension of Goguen's and Meseguer's frameworks for institutions and entailment systems

- The term monad can be abstracted by $\Theta\colon \mathtt{Sign} \to \mathtt{Mnd}[\mathtt{C}]$ with $\mathtt{Mnd}[\mathtt{C}]$ being the category of monads over $\mathtt{C}$ of 'variable objects'.
- Clearly, a special case is $\Theta(\Sigma) = \mathbf{T}_\Sigma$.

- The Sen functor is abstracted as

$$\mathsf{Sen}\colon \mathtt{Mnd}[\mathtt{C}] \to [\mathtt{C}, \mathtt{D}],$$

where $\mathtt{D}$ is monoidal biclosed and $[\mathtt{C}, \mathtt{D}]$ is the functor category, that is, for any monad $\mathbf{F} \in \mathsf{Ob}(\mathtt{Mnd}[\mathtt{C}])$ we have a functor

$$\mathsf{Sen}(\mathbf{F})\colon \mathtt{C} \to \mathtt{D}$$

taking some object of variables to sentences over that object.
- Sen_{HCL} is of the form $\mathsf{Sen}(\mathbf{T}_\Sigma)\colon \mathtt{Set}_\mathtt{S} \to \mathtt{Set}$, where $\Sigma = (S, \Omega)$.
- $\mathsf{Sen}_{HCL}(\mathfrak{Q})$ of the form $\mathsf{Sen}(\mathbf{T}_\Sigma)\colon \mathtt{Set}(\mathfrak{Q})_\mathtt{S} \to \mathtt{Set}(\mathfrak{Q})$ can be constructed.

- $\mathsf{Sen}(\Theta(\Sigma)) \colon \mathtt{C} \to \mathtt{D}$
- $\mathsf{Sen}(\mathbf{T}_{\Sigma}) \colon \mathtt{Set}(\mathfrak{Q})_{\mathtt{S}} \to \mathtt{Set}(\mathfrak{Q})$
- Note how the signature is underlying everything, and once the term functor has been abstracted, substitution is really the "fuel" of logic inference.
- Generalized proof calculus can now be done without explicitly saying what the terms are!
- Soundness and completeness, conceptully generalized, can potentially be analysed in a very general sense, and generalized substitution (for terms, not sentences!) is a key issue in this general framework of *Lative Logic*.

A generalized entailment system, $\mathscr{E}$, is a structure $\mathscr{E} = (\mathtt{Sign}, \mathrm{Sen}, \mathbf{\Phi}, L, \vdash)$ where

- $\mathtt{Sign}$ is a category of signatures;
- Sen is the 'sentence functor';
- $\mathbf{\Phi} = (\Phi, \eta)$ is a premonad over $\mathtt{C}$ with an object of $\Phi\mathrm{Sen}(\Sigma)$ being called a *theoremata*;
- L is a completely distributive lattice; and
- $\vdash$ is a family of L-valued relations consisting of

$$\vdash_\Sigma : \Phi\mathrm{Sen}(\Sigma) \times \Phi\mathrm{Sen}(\Sigma) \to L$$

  for each signature $\Sigma \in \mathrm{Ob}(\mathtt{Sign})$ where $\vdash_\Sigma$ is called a Σ*-entailment*.

These are subject to the condition that, for $\Gamma_1, \Gamma_2, \Gamma_3 \in \Phi\mathsf{Sen}(\Sigma)$ (over $\mathtt{Set}$), each $\vdash_\Sigma$

- is reflexive, that is, $(\Gamma_1 \vdash_\Sigma \Gamma_1) = \top$;
- is *axiom monotone*, that is,

$$((\Gamma_1 \vee \Gamma_2) \vdash_\Sigma \Gamma_3) \geq (\Gamma_1 \vdash_\Sigma \Gamma_3) \vee (\Gamma_2 \vdash_\Sigma \Gamma_3);$$

- is *consequent invariant*, i.e.,

$$(\Gamma_1 \vdash_\Sigma \Gamma_2) \wedge (\Gamma_1 \vdash_\Sigma \Gamma_3) = (\Gamma_1 \vdash_\Sigma (\Gamma_2 \vee \Gamma_3));$$

- is transitive in the sense that

$$(\Gamma_1 \vdash_\Sigma \Gamma_2) \wedge ((\Gamma_1 \vee \Gamma_2) \vdash_\Sigma \Gamma_3) \leq (\Gamma_1 \vdash_\Sigma \Gamma_3); \text{ and}$$

- is an $\vdash$-*translation*, meaning that

$$(\Gamma_1 \vdash_\Sigma \Gamma_2) \leq (\Phi\mathsf{Sen}(\sigma)(\Gamma_1) \vdash_{\Sigma'} \Phi\mathsf{Sen}(\sigma)(\Gamma_2))$$

  for all signature morphisms $\sigma \in \mathsf{Hom}_{\mathtt{Sign}}(\Sigma, \Sigma')$.

A generalized institution

$$\mathscr{I} = (\mathtt{Sign}, \mathrm{Sen}, \mathrm{Mod}, \mathbf{\Phi}, L, \models)$$

is a structure where

- $\mathtt{Sign}$ is a category of signatures;
- Sen is a functor $\mathrm{Sen}\colon \mathtt{Sign} \to \mathtt{Set}$ taking signatures to sentences,
- $\mathrm{Mod}\colon \mathtt{Sign} \to \mathtt{Cat}^{op}$ is a functor with $\mathrm{Mod}(\Sigma)$ representing the category of Σ-*models*;
- L is a completely distributive lattice; and
- $\models$ is a family of L-valued relations consisting of

$$\models_\Sigma\colon\ \mathrm{Ob}(\mathrm{Mod}(\Sigma)) \times \Phi\mathrm{Sen}(\Sigma) \to L$$

  for each signature $\Sigma \in \mathrm{Ob}(\mathtt{Sign})$ where $\models_\Sigma$ is called a Σ-*satisfaction relation*.

The $\models_\Sigma$ relations must fulfill the *satisfaction condition* that states that for all signature morphisms $\sigma \in \mathrm{Hom}_{\mathtt{Sign}}(\Sigma, \Sigma')$, models $M \in \mathrm{Ob}(\mathrm{Mod}(\Sigma))$ and theoremata $\Gamma \in \Phi\mathrm{Sen}(\Sigma)$, $\models_\Sigma$ must be such that

$$(\mathrm{Mod}(\sigma)(M) \models_\Sigma \Gamma) = (M \models_{\Sigma'} \Phi\mathrm{Sen}(\sigma)(\Gamma)).$$

A logic is a tuple

- $\mathscr{L} = (\mathtt{Sign}, \mathtt{C}, \Theta, \mathtt{D}, \mathsf{Sen}, \mathsf{Mod}, \Phi, L, \vdash, \models)$

and an object in a category of logics, generalizing quite broadly the Burstall-Goguen-Meseguer frameworks of institutions and entailment systems. Doing so we in fact more specific about the sentence functor, which in Burstall-Goguen-Meseguer frameworks are concretized only in specific examples such as for FOL and EL.

A logic is an object in a category of logics, where there are morphisms between logics. This is a “formal dialogic” view of logic and dialogue.

Humans use their own structure of natural language and logic, and when communicating, the morphisms transforms what is said by one to be understood by the other.

Type theory as initiated by Schönfinkel, Curry and Church

- As we have seen, going from one-sorted to many-sorted must be done properly, so that going beyond `Set` can be done properly.
- Schönfinkel was 'untyped', Curry 'simply typed', and Church introduced the intuition about his ι and o 'types'.
- They were all unclear about in which signature these 'types' (as sorts) and 'type constructors' (as operators) shold reside.

- The formal description of the conventional set of terms over a signature is clear, but the formalization of the set of λ-terms is less obvious.
- Could we, for instance, avoid the renaming issue with a more strict construction of the set of λ-terms?

- We introduce ‘levels of signatures’ in order to handle the ‘type’ sort (Church’s ι) and type constructors in a signature of its own.
- Further we depart from λ-abstraction in that we say that operators in the underlying signature “owns” their abstractions.
- Note that Church indeed called “λ” an improper symbol.

Levels of signatures for simply typed λ-calculus

1. Level one: The level of 'primitive and underlying' sorts and operations, with a many-sorted signature

$$\Sigma = (S, \Omega)$$

2. Level two: The level of 'type constructors', with a single-sorted signature

$$\lambda_\Sigma = (\{\iota\}, \{\mathtt{s} :\to \iota \mid \mathtt{s} \in S\} \cup \{\Rrightarrow : \iota \times \iota \to \iota\})$$

3. Level three: The level in which we may construct 'λ-terms' based on the signature

$$\Sigma^\lambda = (S^\lambda, \Omega^\lambda)$$

where $S^\lambda = \mathsf{T}_{\lambda_\Sigma}\emptyset$, $\Omega^\lambda = \{\omega^\lambda_{i_1,\ldots,i_n} :\to (\mathtt{s}_{i_1} \Rrightarrow \cdots \Rrightarrow (\mathtt{s}_{i_{n-1}} \Rrightarrow (\mathtt{s}_{i_n} \Rrightarrow \mathtt{s})\cdots) \mid \omega : \mathtt{s}_1 \times \ldots \times \mathtt{s}_n \to \mathtt{s} \in \Omega, (i_1, \ldots, i_n)$ is a permutation of $(1, \ldots, n)\} \cup \{\mathtt{app}_{\mathtt{s},\mathtt{t}} : (\mathtt{s} \Rrightarrow \mathtt{t}) \times \mathtt{s} \to \mathtt{t}\}$

The natural numbers signature in levels

1. Level one:

$$\mathtt{NAT} = (\{\mathtt{nat}\}, \{0 :\to \mathtt{nat}, \mathtt{succ} : \mathtt{nat} \to \mathtt{nat}\})$$

2. Level two:

$$\lambda_{\mathtt{NAT}} = (\{\iota\}, \{\mathtt{nat} :\to \iota, \Rrightarrow : \iota \times \iota \to \iota\})$$

3. Level three:

$$\Sigma^{\lambda} = (\mathsf{T}_{\lambda_{\mathtt{NAT}}\emptyset}, \Omega^{\lambda})$$

where $\Omega^{\lambda} = \{0^{\lambda} :\to \mathtt{nat}, \mathtt{succ}_1^{\lambda} :\to (\mathtt{nat} \Rrightarrow \mathtt{nat})\} \cup \{\mathtt{app}_{\mathtt{s},\mathtt{t}} : (\mathtt{s} \Rrightarrow \mathtt{t}) \times \mathtt{s} \to \mathtt{t}\}$

λ-calculus

... so then we can do λ-calculus, fuzzy λ-calculus, λ-calculus with fuzzy, and so on.

$\Sigma_{\mathtt{DescriptionLogic}} = (S, \Omega)$

1. $S = \{\mathtt{concept}\}$, and we may add constants like $\mathtt{c}_1, \ldots, \mathtt{c}_n :\rightarrow \mathtt{concept}$.
2. We include a type constructor $\mathtt{P} : \mathtt{type} \rightarrow \mathtt{type}$ into S_Ω, with an intuitive semantics of being the powerset functor, so that $\mathtt{Pconcept}$ is the constructed type for "powerconcept".
3. "Roles" are $\mathtt{r} :\rightarrow (\mathtt{Pconcept} \Rrightarrow \mathtt{PPconcept})$, and we need operators $\eta :\rightarrow (\mathtt{concept} \Rrightarrow \mathtt{Pconcept})$ and $\mu :\rightarrow (\mathtt{PPconcept} \Rrightarrow \mathtt{Pconcept})$ in Ω', so that "$\exists r.x$" can be defined as

$$\mathrm{app}_{\mathtt{PPconcept},\mathtt{Pconcept}}(\mu, \mathrm{app}_{\mathtt{Pconcept},\mathtt{PPconcept}}(\mathtt{r}, x)).$$

Renaming

- In traditional notation, substituting x by $\mathtt{succ}(y)$ in $\lambda y.\mathtt{succ}(x)$ should cause a rename of the bound variable y, e.g., $\lambda z.\mathtt{succ}(\mathtt{succ}(y))$.
- On level 1, we have the substitution (Kleisli morphism) $\sigma_{\mathtt{nat}} : X_{\mathtt{nat}} \to \mathsf{T}_{\mathtt{NAT},\mathtt{nat}}\{X_{\mathtt{t}}\}_{\mathtt{t}\in\{\mathtt{nat}\}}$, where $\sigma_{\mathtt{nat}}(x) = \mathtt{succ}(y)$, x being a variable on level 1, and the extension of $\sigma_{\mathtt{nat}}$ is $\mu_{\mathtt{nat}} \circ \mathsf{T}_{\mathtt{NAT},\mathtt{nat}}\sigma_{\mathtt{nat}} : \mathsf{T}_{\mathtt{NAT},\mathtt{nat}}\{X_{\mathtt{t}}\}_{\mathtt{t}\in\{\mathtt{nat}\}} \to \mathsf{T}_{\mathtt{NAT},\mathtt{nat}}\{X_{\mathtt{t}}\}_{\mathtt{t}\in\{\mathtt{nat}\}}$.
- On level 3 we have $\sigma'_{\mathtt{nat}} : X_{\mathtt{nat}} \to \mathsf{T}_{\mathtt{NAT}',\mathtt{nat}}\{X_{\mathtt{t}}\}_{\mathtt{t}\in S''}$, with $\sigma'_{\mathtt{nat}}(x) = \mathrm{app}_{\mathtt{nat},\mathtt{nat}}(\mathtt{succ}_1^{\lambda}, x)$, x being a variable on level 3, and $\mu'_{\mathtt{nat}} \circ \mathsf{T}_{\mathtt{NAT}',\mathtt{nat}}\sigma'_{\mathtt{nat}}(\mathrm{app}_{\mathtt{nat},\mathtt{nat}}(\mathtt{succ}_1^{\lambda}, x))$ requiring no renaming.

Schönfinkel's *Bausteine* (1920)

The *constancy function* C, defined as $(Ca)y = a$, can be seen as the type constructor $\mathtt{C} : \mathtt{type} \times \mathtt{type} \to \mathtt{type}$ fulfilling the 'equational condition' $\mathtt{C}(\mathtt{s}, \mathtt{t}) = \mathtt{s}$, and $\mathfrak{A}_{\mathtt{C}_\Sigma}$ would again be a functor fulfilling the corresponding criteria. Additionally, C can also be seen as an operator within Σ' as $\mathtt{C}_{\mathtt{s},\mathtt{t}} :\to (\mathtt{s} \Rrightarrow (\mathtt{t} \Rrightarrow \mathtt{s}))$, with $\mathfrak{A}_{\Sigma'}(\mathtt{C}_{\mathtt{s},\mathtt{t}}) \in \mathsf{Hom}(\mathfrak{A}_{\Sigma'}(\mathtt{s}), \mathsf{Hom}(\mathfrak{A}_{\Sigma'}(\mathtt{t}), \mathfrak{A}_{\Sigma'}(\mathtt{s})))$ so that $\mathfrak{A}_{\Sigma'}(\mathtt{C}_{\mathtt{s},\mathtt{t}})(x)(y) = x$ for $x \in \mathfrak{A}_{\Sigma'}(\mathtt{s})$ and $y \in \mathfrak{A}_{\Sigma'}(\mathtt{t})$. A sentence, in equational type logic, prescribing the constancy function condition would then look like $\mathtt{app}_{\mathtt{s},\mathtt{t}}(\mathtt{C}_{\mathtt{s},\mathtt{t}}, \mathtt{t}) = \mathtt{s}$.

- Some of Schönfinkel's "operators" I, C, T, Z and S can be 'simply typed' on level two and three (I, C), and some on level three only (T, Z and S).
- See "Modern eyes on λ-calculus" (GLIOC notes, www.glioc.com)

Curry's *functionality* (1934)

Curry, like Schönfinkel, is weak on making distinction between syntax and semantics, so F on signature level two would be $F = \Rrightarrow : \texttt{type} \to \texttt{type}$ so that FXY is the term $X \Rrightarrow Y$, with $X, Y :: \texttt{type}$. Thus, Curry's $\vdash FXYf$, *representing the statement that* f *belongs to that category*, means f is the constant $f : X \Rrightarrow Y$. Both F and f is by Curry called 'entities', but they are operators within different signatures.

- Curry believes that point that variables *may be introduced into the formal developments without loss of precision*.
- This, in our view, is the "what belongs and what does nt" of variables, leading to fear about 'loss of precision'.
- Variables were at that time mostly viewed as 'distinct from constants'.
- Curry writes further that *variables are not the names of any entities whatever, but are "incomplete symbols", whose function is to indicate possibilities of substitution*.

Church's *simple typing* (1940)

We purposely refrain from making more definite the nature of the types o and ι, the formal theory admitting of a variety of interpretations in this regard. Of course the matter of interpretation is in any case irrelevant to the abstract construction of the theory, and indeed other and quite different interpretations are possible (formal consistency assumed).

- Our $(\beta \Rrightarrow \alpha)$ is Church's $(\beta\alpha)$.
- Speaking in terms of modern type theory involving '$\mathtt{type}$' and '$\mathtt{prop}$', Church's ι, as we have said, is our $\mathtt{type}$ on signature level two, but o is not something like $\mathtt{bool}$, but more like a '$\mathtt{prop}$', which is more unclear.
- We could imagine a $\Rrightarrow_{\mathtt{prop},\mathtt{type},\mathtt{type}} : \mathtt{type} \times \mathtt{type} \to \mathtt{prop}$ corresponding to Church's $o\iota\iota$, but it is not obvious how to deal with it.
- Intuitively, a quantifier may look like $\Pi : \mathtt{type} \times \mathtt{prop} \to \mathtt{prop}$, i.e., like Church's $\Pi_{o(o\alpha)}$, but again, it is not clear how to proceed.
- The algebras of $\mathtt{type}$ and $\mathtt{prop}$ also need to be settled.

- Church's $I_{\alpha\alpha}$ operator is Schönfinkel's identity function I, and Church's $K_{\alpha\beta\alpha}$ operator is Schönfinkel's constancy function C.
- His syntactic definitions of natural numbers $0_{\alpha'}$, $1_{\alpha'}$, $2_{\alpha'}$, $3_{\alpha'}$, etc., is then kind of assuming that the topmost signature Σ is the empty signature.
- Church's 'variable binding' operator, or *choice function*, $\iota_{\alpha(o\alpha)}$, is influence e.g. by Hilbert's ϵ-operator in the ϵ-calculus culminating in Ackermann's thesis 1924.
- The $\iota_{\alpha(o\alpha)}$ operator obviously has its counterpart in our framework as well, but appears differently since variables are only implicitly pointed at by the indices appearing in $\omega^{\lambda}_{i_1,\ldots,i_n}$.

The Brouwer-Heyting-Kolmogorov interpretation

Appears in its well-known form propositionally presented by Komogorov in 1932, *Zur Deutung der Intuitionistischen Logik*:

- Es gilt dann die folgende merkwürdige Tatsache: *Nach der Form fällt die Aufgabenrechnung mit der Brouwersehen, von Herrn Heyting neuerdings formaliaierten, intuitionistischen Logik zusammen*.
- Wit glauben, daß nach diesen Beispielen und Erklärungen die Begriffe “Aufgabe” und “Lösung der Aufgabe” in allen Fällen, welche in den konkreten Gebieten der Mathematik vorkommen, ohne Mißverständnis gebraucht werden können. Die Hauptbegriffe der Aussagenlogik “Aussage” und “Beweis der Aussage” befinden sich nicht in besserer Lage.
- Wenn a und b zwei Aufgaben sind, bezeichnet $a \wedge b$ die Aufgabe “beide Aufgaben a und b lösen”, ...

The Curry-Howard isomorphism

Appears in its most well-known form presented by Howard in 1969/1980, *The formulae-as-types notion of construction* and was based e.g. on Curry's and Fey's *Combinatory Logic* from 1958:

- The following consists of notes which were privately circulated in 1969. Since they have been referred to a few times in the literature, it seems worth while to publish them. (Howard,1980)
- Let $P(\supset)$ denote positive implicational propositional logic. By a type symbol is meant a formula of $P(\supset)$. (Howard,1980)
- This can be seen as $\Sigma = (S, \emptyset)$, on level 1, where S is viewed as the set of 'prime formulae', $\mathsf{T}_{\lambda_\Sigma}\emptyset$ is the set of all formulae in $P(\supset)$.

- If we now have $\mathtt{BOOL} = (\{\mathtt{bool}\}, \{\mathsf{a}_i :\rightarrow \mathtt{bool} \mid i \in I\} \cup \{\Rightarrow, \wedge : \mathtt{bool} \times \mathtt{bool} \rightarrow \mathtt{bool}\})$ on level one, then $\mathtt{BOOL}' = (\mathsf{T}_{\lambda_\Sigma}\emptyset, \{\mathsf{a}_{i_0}^{\lambda} :\rightarrow \mathtt{bool} \mid i \in I\} \cup \{\Rightarrow_{1,2}^{\lambda}, \wedge_{1,2}^{\lambda} :\rightarrow (\mathtt{bool} \Rrightarrow (\mathtt{bool} \Rrightarrow \mathtt{bool}))\} \cup \{\mathtt{app}_{\mathtt{s},\mathtt{t}} : (\mathtt{s} \Rrightarrow \mathtt{t}) \times \mathtt{s} \rightarrow \mathtt{t} \mid \mathtt{s}, \mathtt{t} \in \mathsf{T}_{\lambda_\Sigma}\emptyset\})$ providing $\mathsf{T}_{\mathtt{BOOL}'}\emptyset$ on level three is not to be confused with $\mathsf{T}_{\lambda_\Sigma}\emptyset$ on level two.
- Adding Schönfinkel's $\mathtt{C}_{\mathtt{s},\mathtt{t}} :\rightarrow (\mathtt{s} \Rrightarrow (\mathtt{t} \Rrightarrow \mathtt{s}))$ (Curry's $\mathtt{K}$) as an operator on level 3 is then seen as an ‘axiom’.

Algebras

- In the two-valued case, $\mathfrak{A}(\texttt{bool})$ is often $\{\mathit{false}, \mathit{true}\}$, so that $\mathfrak{A}(\texttt{F}) = \mathit{false}$ and $\mathfrak{A}(\texttt{T}) = \mathit{true}$.
- $\mathfrak{A}(\&) : \mathfrak{A}(\texttt{bool}) \times \mathfrak{A}(\texttt{bool}) \to \mathfrak{A}(\texttt{bool})$, is expected to be defined by the usual 'truth table'.
- We may assign for a signature $\Sigma_{PL} = (S_{PL}, \Omega_{PL})$ a pair, the 'many-sorted algebra', $(\mathsf{T}_{\Sigma_{PL}} X_S, (\mathfrak{A}(\omega))_{\omega \in \Omega_{PL}})$, where $X_{\mathtt{s}} = \varnothing$ if $\mathtt{s} \neq \mathtt{bool}$.
- Then, $(\bigcup_{\mathtt{s} \in S} (\mathrm{arg}^{\mathtt{s}} \circ \mathsf{T}_{\Sigma_{PL}}) X_S, (\texttt{F}, \texttt{T}, \&, \neg))$ serves as a traditional Boolean algebra, when certain equational laws are given.

Programs and their interpretations

- $\Gamma = \{(h_1, b_1), \ldots, (h_n, b_n)\} \subseteq \mathsf{Sen}_{HCL} X_S$
- $(U_\Gamma)_S = \mathsf{T}_\Sigma \emptyset_S = (\mathsf{T}_{\Sigma,\mathtt{s}} \emptyset_S)_{\mathtt{s} \in S}$
- $\bigcup_{\mathtt{s} \in S} (U_\Gamma)_\mathtt{s}$ corresponds to the traditional and unsorted view of the *Herbrand universe*
- $B_\Gamma = (\mathrm{arg}^{\mathtt{bool}} \circ \mathsf{T}_{\Sigma_{PL \setminus \neg, \&}} \circ \mathsf{T}_\Sigma)\ \emptyset_S$ corresponds to the *Herbrand base*
- Herbrand interpretations of a program Γ are subsets $\mathcal{I} \subseteq B_\Gamma$
- we also need what we call the *Herbrand expression base*: $B_\Gamma^\& = (\mathrm{arg}^{\mathtt{bool}} \circ \mathsf{T}_{\Sigma_{PL \setminus \neg}} \circ \mathsf{T}_\Sigma)\ \emptyset_S$
- a Herbrand interpretation $\mathcal{I}$ canonically extends to a *Herbrand expression interpretation* $\mathcal{I}^\& \subseteq B_\Gamma^\&$

Substitution fuzzy Horn clause logic

- fuzzy sets of predicates: $\mathsf{L}B_\Gamma = (\mathsf{L} \circ \mathrm{arg}^{\mathtt{bool}} \circ \mathsf{T}_{\Sigma_{PL \setminus \neg, \&}} \circ \mathsf{T}_\Sigma)\ \emptyset_S$
- sentence functor:
  $\mathrm{Sen}_{SFHCL} = (\mathrm{arg}^{\mathtt{bool}})^2 \circ ((\mathsf{T}_{\Sigma_{PL \setminus \neg, \&}} \times \mathsf{T}_{\Sigma_{PL \setminus \neg}}) \circ \mathsf{L}_S \circ \mathsf{T}_\Sigma \circ \phi^{S \setminus \mathtt{bool}})$
- ground predicates over fuzzy sets of terms:
  $B_\Gamma^{\mathsf{L}} = (\mathrm{arg}^{\mathtt{bool}} \circ \mathsf{T}_{\Sigma_{PL \setminus \neg, \&}} \circ \mathsf{L}_S \circ \mathsf{T}_\Sigma)\ \emptyset_S$
- the fuzzy sets of ground predicates is enabled by the 'swapper': $\varsigma : \mathsf{T}_{\Sigma_{PL \setminus \neg, \&}} \circ \mathsf{L}_S \to \mathsf{L}_S \circ \mathsf{T}_{\Sigma_{PL \setminus \neg, \&}}$

Fixpoints

- considering the effect of substitutions with fuzzy sets of terms: $\varpi^{\mathsf{L}} : \mathsf{L}B_{\Gamma}^{\mathsf{L}} \to \mathsf{L}B_{\Gamma}^{\mathsf{L}}$
- $\mathrm{arg}^{\texttt{bool}}\varsigma_{\mathsf{T}_{\Sigma}\emptyset_S} : B_{\Gamma}^{\mathsf{L}} \to \mathsf{L}B_{\Gamma}$

$$\varpi^{\mathsf{L}}(\mathcal{I})(\sigma_{\texttt{bool}}^{\mathsf{L},head}(h)) = \\ (\bigvee_{t \in B_{\Gamma}}(\mathrm{arg}^{\texttt{bool}}\varsigma_{\mathsf{T}_{\Sigma}\emptyset_S}(h))(t)) \wedge \mathcal{I}^{\mathsf{L},\&}(\sigma_{\texttt{bool}}^{\mathsf{L},body}(b))$$

Terminologies, classifications and ontologies in social and health care

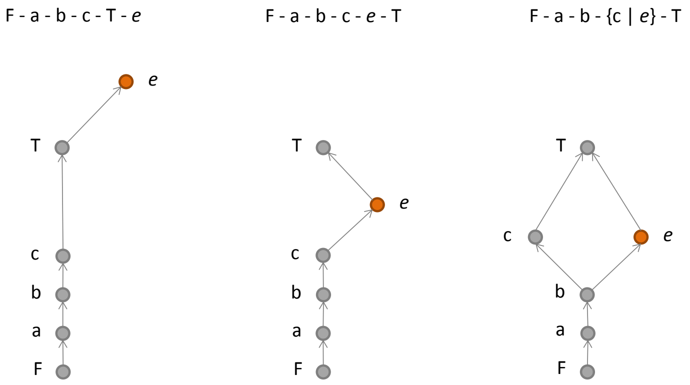
- WHO's ICF and ICD-10
- ATC for drugs
- SNOMED which is believed to have description logic as its underlying logic for ontology (health onttology and web ontology is not the same thing!)
- fall risk and fall injury risk

```
Muscle functions (ICF b730-b749)
  Muscle power functions (b730)
    ...
    Power of muscles of all limbs (b7304)
    ...
  Muscle tone functions (b735)
  Muscle endurance functions (b740)
```

The ICF datatypes and its generic scale of quantifiers:

```
xxx.0 NO problem          (none, absent, ...)
xxx.1 MILD problem        (slight, low, ...)
xxx.2 MODERATE problem    (medium, fair, ...)
xxx.3 SEVERE problem      (high, extreme, ...)
xxx.4 COMPLETE problem    (total, ...)
xxx.8 not specified
xxx.9 not applicable
```

Unknown as unital *e* with 5-valued set {F, a, b, c, T} of **truth values**, corresponding to the ICF valuations,
including the unknown as 'not specified' (problem qualifier code 8)



ICD-10

```
S52   fracture of forearm
S52.5 fracture of lower end of radius
```

and conflicting ICD-10 extensions, with the ICD-10-CM adopted in the US going further in direction of

```
S52.53   Colles' fracture of radius
S52.532  Colles' fracture of left radius
S52.532D Colles' fracture of left radius,
         subsequent encounter for closed
         fracture with routine healing
```

where “3” for ‘Colles’ means dorsal displacement, “2” and “-” after “53” means ‘left or unspecified arm, and “D” means subsequent encounter for closed fracture with routine healing.

For comparison, in Germany, the ICD-10-GM (2014) uses

```
S52.5  Distale Fraktur des Radius
S52.51 Extensionsfraktur, Colles-Fraktur
```

i.e.,'Colles' now is "51", where the US version says "53". Thus, we have to be "internationally careful" when we see a code like "S52.51".
In Sweden, the ICD-10-SE is only ICD

```
S52.5 Fraktur på nedre delen av radius
```

whereas the Swedish Orthopaedic Association uses

```
S52.50/51 Distal radius (Barton, Colles, Smith)
```

where "0" is left and "1" is right, so the Swedish "S52.51" is different from the German one, and different from the corresponding US code.

Sleeping pills affect the balance so the use of sedatives is a fall risk factor

Anatomic Therapeutic Chemical (ATC) classification of *nitrazepam* (code C08DA01), long-acting drug for insomnia:

N	nervous system	1st level main anatomical group
N05	psycholeptics	2nd level, therapeutic subgroup
N05C	hypnotics and sedatives	3rd level, pharmacological subgroup
N05CD	benzodiazepine derivatives	4th level, chemical subgroup
N05CD02	nitrazepam	5th level

Downton's Fall Risk Index (DFRI) assessment scale includes the item 'tranquilizers/sedatives' under "Medications", so the user is providing drug information related to a pharmacological subgroup (3rd level), where nitrazepam (5th level) is one of the most fall-risk-increasing drugs (FRIDs). Then again, on interventions it is easy to speak generally about the effect of "withdrawal of psychotropics" (2nd level). Obviously, from formal information management point of view, the health care domain does not always consider data typing and granularity issues.

For ATC, on level two we could have

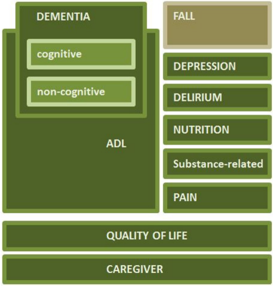
$$\mathtt{1st}, \mathtt{2nd}, \mathtt{3rd}, \mathtt{4th}, \mathtt{5th} :\to \mathtt{type}$$

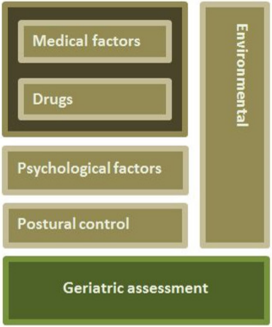
and on level three

$$\begin{aligned}
\mathtt{PharmacologicIntervention} &:\to \mathsf{P}(\mathtt{3rd}) \\
\mathtt{DrugPrescriptions} &:\to \mathsf{P}(\mathtt{5th}) \\
\textit{hypnotics_and_sedatives} &:\to \mathtt{3rd} \\
\textit{benzodiazepine_derivatives} &:\to \mathtt{4th} \\
\textit{nitrazepam} &:\to \mathtt{5th} \\
\textit{drug} &:\to \mathtt{5th} \\
\phi^{\mathtt{5th}\to\mathtt{4th}} &: \mathtt{5th} \to \mathtt{4th} \\
\phi^{\mathtt{4th}\to\mathtt{3rd}} &: \mathtt{4th} \to \mathtt{3rd} \\
\phi^{\mathtt{5th}\to\mathtt{3rd}} &: \mathtt{5th} \to \mathtt{3rd}
\end{aligned}$$

This then makes a clear distinction between *nitrazepam* as a term of type $\mathtt{5th}$ and $\phi^{\mathtt{5th}\rightarrow\mathtt{3rd}}(\textit{nitrazepam})$ as a sedative of type $\mathtt{3rd}$. Further, for the variable *drug*, we can make a substitution with *nitrazepam*, because the types match, but we cannot substitute with *hypnotics_and_sedatives*. For Downton's index the consequence is that $\phi^{\mathtt{5th}\rightarrow\mathtt{3rd}}(\textit{drug})$ may appear as a value in the scale, but not *drug*. This is also important in considerations of uncertainty. A relative to a patient may be fairly sure about *hypnotics_and_sedatives*, but not all that certain about that sedative being a *benzodiazepine_derivatives*. Additional operators is required to capture the notion of uncertainty being carried over between ATC levels.

Gerontological and geriatric assessment in general, and fall risk assessment in particular.





Implementations e.g. within the AAL Call 4 project AiB (Ageing in Balance)

Level one:

$$\texttt{GERONTIUM} = (S, \Omega)$$

where $S = \{\texttt{nat}, \texttt{bool}, \texttt{scale}, \ldots\}$. Operators in Ω can be provided in a number of ways, and is left unspecified at this point.

Level two:

$$\lambda_{\texttt{GERONTIUM}} = (\{\texttt{Observation}, \texttt{Assessment}\}, \lambda_\Omega)$$

λ_Ω:

$$\begin{aligned}
\texttt{s} &: \to \texttt{Observation}, \texttt{s} \in S \\
\boxtimes &: \texttt{Observation} \times \texttt{Observation} \to \texttt{Observation} \\
\boxplus &: \texttt{Assessment} \times \texttt{Assessment} \to \texttt{Assessment} \\
\texttt{P} &: \texttt{Assessment} \to \texttt{Assessment} \\
\Rrightarrow_{\texttt{Observation}} &: \texttt{Observation} \times \texttt{Observation} \to \texttt{Observation} \\
\Rrightarrow_{\texttt{Assessment}} &: \texttt{Assessment} \times \texttt{Assessment} \to \texttt{Assessment}
\end{aligned}$$

$$\begin{aligned}
\texttt{CognitiveDementia} &: \to \texttt{Assessment} \\
\texttt{Non-CognitiveDementia} &: \to \texttt{Assessment} \\
\texttt{ADL} &: \to \texttt{Assessment} \\
\texttt{Depression} &: \to \texttt{Assessment} \\
\texttt{Delirium} &: \to \texttt{Assessment} \\
\texttt{Nutrition} &: \to \texttt{Assessment} \\
\texttt{SubstanceRelated} &: \to \texttt{Assessment} \\
\texttt{Pain} &: \to \texttt{Assessment}
\end{aligned}$$

$$\texttt{GeriatricAssessment} : \to \texttt{Assessment}$$

$$
\begin{aligned}
\texttt{MedicalFactors} &: \to \texttt{Assessment}\\
\texttt{Drugs} &: \to \texttt{Assessment}\\
\texttt{PsychologicalFactors} &: \to \texttt{Assessment}\\
\texttt{PosturalControl} &: \to \texttt{Assessment}\\
\texttt{EnvironmentalFactors} &: \to \texttt{Assessment}
\end{aligned}
$$

$$
\texttt{FallRiskAssessment} : \to \texttt{Assessment}
$$

Level three:

$$\mathtt{GERONTIUM}^{\lambda} = (\mathsf{T}_{\lambda_{\mathtt{GERONTIUM}}\emptyset}, \Omega^{\lambda})$$

Ω^{λ}, including the *Falls Efficacy Scale - International* (FES-I) as an example of an assessment scale:

$$\begin{aligned}\mathtt{FES{-}I} : \to (&\mathtt{scale4}\natural^{16}\\ &\Rrightarrow (\mathtt{scale64} \boxtimes \mathtt{scale3}\\ &\boxtimes \mathtt{PsychologicalFactors}))\end{aligned}$$

$$\mathtt{Odepression} : \mathtt{P}\ \mathtt{Depression} \to \mathtt{Depression}$$

$$\begin{aligned}\mathtt{OA} &: \to \mathtt{P}\ \mathtt{CognitiveDementia} \boxplus \ldots\\ \mathtt{FallOA} &: \to \mathtt{P}\ \mathtt{MedicalFactors} \boxplus \ldots\end{aligned}$$

$$\mathtt{app}_{\mathtt{s},\mathtt{t}} : (\mathtt{s} \Rrightarrow \mathtt{t}) \times \mathtt{s} \to \mathtt{t}$$

Mille grazie!

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