Partially ordered monads and powerset Kleene algebras

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Abstract. Monads are used for various applications in computer science, and well-known is e.g. the interpretation of morphisms in the Kleisli category of the term monad as variable substitutions assigning variables to terms. An application building upon this observation is the equivalence between most general unifiers and co-equalizers in this category [13]. In this paper we will use monads with additional structure, namely partially ordered monads. We show how partially ordered monads can be used in order to obtain a generalised notion of Kleene powerset algebras building upon more general powerset functor settings beyond strings [11] and relations [14].

1 Introduction

Monads equipped with order structures extends suitably to so called partially ordered monads. In this paper we will show how these partially ordered monads, together with their subconstructions, contribute to providing a generalised notion of powerset Kleene algebras. This generalisation builds upon more general powerset functor setting far beyond just strings [11] and relations [14].

Previous work on monadic instrumentation for various many-valued set functors include investigations as initiated e.g. in [1, 2]. Monads were used in particular for compactifications of generalised convergence spaces based on double powerset monads. Further work involving partially ordered monads are found in [8, 9].

The present paper is organised as follows. Section 2 provides notations for and examples of monads and corresponding Kleisli categories. Section 3 provides extensions to partially ordered monads, and Section 4 introduces their subconstructions together with important examples for powerset Kleene algebras as introduced in Section 5.

2 Monads

Let C be a category. A monad (or triple, or algebraic theory) over C is written as $\mathbf{F} = (F, \eta, \mu)$, where $F : \mathsf{C} \to \mathsf{C}$ is a (covariant) functor, and $\eta : id \to F$

and $\mu : F \circ F \to F$ are natural transformations for which $\mu \circ F\mu = \mu \circ \mu F$ and $\mu \circ F\eta = \mu \circ \eta F = id_F$ hold. A Kleisli category $\mathsf{C}_{\mathbf{F}}$ for a monad $\mathbf{F}$ over a category C is given with objects in $\mathsf{C}_{\mathbf{F}}$ being the same as in C, and morphisms being defined as $hom_{\mathsf{C}_{\mathbf{F}}}(X,Y) = hom_{\mathsf{C}}(X, FY)$. Morphisms $f: X \rightharpoondown Y$ in $\mathsf{C}_{\mathbf{F}}$ are thus morphisms $f: X \to FY$ in C, with $\eta^F_X: X \to FX$ being the identity morphism.

Composition of morphisms in $\mathsf{C}_{\mathbf{F}}$ is defined as

$$(X \overset{f}{\rightharpoondown} Y) \diamond (Y \overset{g}{\rightharpoondown} Z) = X \overset{\mu^F_Z \circ Fg \circ f}{\longrightarrow} FZ. \tag{1}$$

2.1 The powerset monad

Let L be a completely distributive lattice. For $L = \{0,1\}$ we write $L = 2$. The covariant powerset functor L_{id} is obtained by $L_{id}X = L^X$, i.e. the set of mappings (or L-fuzzy sets) $A : X \to L$, and following [10], for a morphism $f: X \to Y$ in SET, by defining

$$L_{id}f(A)(y) = \bigvee_{f(x)=y} A(x).$$

Further, define $\eta_X : X \to L_{id}X$ by

$$\eta_X(x)(x') = \begin{cases} 1 & \text{if } x = x' \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

and $\mu : L_{id} \circ L_{id} \to L_{id}$ by

$$\mu_X(\mathcal{M})(x) = \bigvee_{A \in L_{id}X} A(x) \wedge \mathcal{M}(A).$$

It was shown in [12] that $\mathbf{L}_{id} = (L_{id}, \eta, \mu)$ indeed is a monad. Note that $\mathbf{2}_{id}$ is the usual covariant powerset monad $\mathbf{P} = (P, \eta, \mu)$, where PX is the set of subsets of X, $\eta_X(x) = \{x\}$ and $\mu_X(\mathcal{B}) = \bigcup \mathcal{B}$.

Further, note that the transitivity condition, relationally viewed as $f \circ f \subseteq f$, translates to $\bigcup Pf(f(x)) \subseteq f(x)$ for all $x \in X$.

Remark 1. The category of 'sets and relations', i.e. where objects are sets and morphisms $f : X \to Y$ are ordinary relations $f \subseteq X \times Y$ with composition of morphisms being relational composition, is isomorphic to the Kleisli category $\mathsf{SET}_{\mathbf{2}_{id}}$. Indeed, relations $f \subseteq X \times Y$ are morphisms $f : X \rightharpoondown Y$ in $\mathsf{SET}_{\mathbf{2}_{id}}$, i.e. morphisms $f : X \to PY$ in SET, and relational composition corresponds exactly to composition according to (1).

Remark 2. Extending functors to monads is not trivial, and unexpected situations may arise. Let the id^2 functor be extended to a monad with $\eta_X(x) = (x,x)$ and $\mu_X((x_1,x_2),(x_3,x_4)) = (x_1,x_4)$. Further, the proper powerset functor P_0, where $P_0X = PX \setminus \{\emptyset\}$, as well as $id^2 \circ P_0$ can, respectively, be extended to monads, even uniquely. However, as shown in [1], $P_0 \circ id^2$ cannot be extended to a monad.

Remark 3. The interaction between monads and algebras is well-known. The tutorial example is the isomorphism between the Kleisli category of the powerset monad and the category of 'sets and relations'. The Eilenberg-Moore category of the powerset monad is isomorphic to the category of complete lattices and join-preserving maps. The Kleisli category of the term monad coincides with its Eilenberg-Moore category and is isomorphic to the category of Ω-algebras. A rather intrepid example, although still folklore, is the isomorphism between the Eilenberg-Moore category of the ultrafilter monad and the category of compact Hausdorff spaces. Here is where "algebra and topology meet".

2.2 The term monad

Notations in this part follow [6], which were adopted also in [1, 5].

Let $\Omega = \bigcup_{n=0}^{\infty} \Omega_n$ be an operator domain, where Ω_n contains n-ary operators. The term functor $T_\Omega\colon \mathsf{SET} \to \mathsf{SET}$ is given as $T_\Omega(X) = \bigcup_{k=0}^{\infty} T_\Omega^k(X)$, where

$$\begin{aligned} T_\Omega^0(X) &= X, \\ T_\Omega^{k+1}(X) &= \{(n, \omega, (m_i)_{i\le n}) \mid \omega \in \Omega_n, n \in N, m_i \in T_\Omega^k(X)\}. \end{aligned}$$

In our context, due to constructions related to generalised terms [3–5], it is more convenient to write terms as $(n, \omega, (x_i)_{i\le n})$ instead the more common $\omega(x_1, \ldots, x_n)$.

It is clear that $(T_\Omega X, (\sigma_\omega)_{\omega\in\Omega})$ is an Ω-algebra, if $\sigma_\omega((m_i)_{i\le n}) = (n, \omega, (m_i)_{i\le n})$ for $\omega \in \Omega_n$ and $m_i \in T_\Omega X$. Morphisms $X \xrightarrow{f} Y$ in $\mathtt{Set}$ are extended in the usual way to the corresponding Ω-homomorphisms $(T_\Omega X, (\sigma_\omega)_{\omega\in\Omega}) \xrightarrow{T_\Omega f} (T_\Omega Y, (\tau_\omega)_{\omega\in\Omega})$, where $T_\Omega f$ is given as the Ω-extension of $X \xrightarrow{f} Y \hookrightarrow T_\Omega Y$ associated to $(T_\Omega Y, (\tau_{n\omega})_{(n,\omega)\in\Omega})$.

To obtain the term monad, define $\eta_X^{T_\Omega}(x) = x$, and let $\mu_X^{T_\Omega} = id_{T_\Omega X}^{\star}$ be the Ω-extension of $id_{T_\Omega X}$ with respect to $(T_\Omega X, (\sigma_{n\omega})_{(n,\omega)\in\Omega})$.

Proposition 1 ([12]). $\mathbf{T}_\Omega = (T_\Omega, \eta^{T_\Omega}, \mu^{T_\Omega})$ *is a monad.*

3 Basic triples and partially ordered monads

Let acSLAT be the category of almost complete semilattices, i.e. partially ordered sets $(X, \le)$ such that the suprema $\sup \mathcal{M}$ of all non-empty subsets $\mathcal{M}$ of X exists. Morphisms $f : (X, \le) \to (Y, \le)$ satisfy $f(\sup \mathcal{M}) = \sup f[\mathcal{M}]$ for non-empty subsets $\mathcal{M}$ of X.

A *basic triple* ([7]) is a triple $\Phi = (\varphi, \le, \eta)$, where $(\varphi, \le) : \mathsf{SET} \to \mathsf{acSLAT}$, $X \mapsto (\varphi X, \le)$ is a covariant functor, with $\varphi : \mathsf{SET} \to \mathsf{SET}$ as the underlying set functor, and $\eta : \mathrm{id} \to \varphi$ is a natural transformation.

If $(\varphi, \le, \eta^\varphi)$ and $(\psi, \le, \eta^\psi)$ are basic triples, then also $(\varphi \circ \psi, \le, \eta^\varphi\psi \circ \eta^\psi)$ is a basic triple.

Example 1. Consider L_{id} as a functor from SET to acSLAT with $\alpha \leq \beta$, $\alpha, \beta \in L_{id}X$, meaning $\alpha(x) \leq \beta(x)$ for all $x \in X$. Then $(L_{id}, \leq, \eta)$ is a basic triple where $\eta_X : X \to L_{id}X$ is given by (2).

Example 2. Consider T_Ω similarly as a functor from SET to acSLAT, with $t_1 \leq t_2$ whenever t_1 is a subterm of t_2 in the usual sense. Then $(T_\Omega, \leq, \eta)$ is a basic triple where $\eta_X : X \to T_\Omega X$ is given as in subsection 2.2.

A *partially ordered monad* is a quadruple $\Phi = (\varphi, \leq, \eta, \mu)$, such that

(i) $(\varphi, \leq, \eta)$ is a basic triple.
(ii) $\mu : \varphi\varphi \to \varphi$ is a natural transformation such that (φ, η, μ) is a monad.
(iii) For all mappings $f, g : Y \to \varphi X$, $f \leq g$ implies $\mu_X \circ \varphi f \leq \mu_X \circ \varphi g$, where $\leq$ is defined argumentwise with respect to the partial ordering of φX.
(iv) For each set X, $\mu_X : (\varphi\varphi X), \leq) \to (\varphi X, \leq)$ preserves non-empty suprema.

Example 3. The basic triples $(L_{id}, \leq, \eta^{L_{id}})$ and $(T_\Omega, \leq, \eta^{T_\Omega})$ can both be extended to partially ordered monads using multiplications μ as given in Section 2.

4 Partially ordered submonads

A *basic subtriple* of Φ is a basic triple $\Phi' = (\varphi', \leq, \eta')$ such that φ' is a subfunctor[3] of φ, $(\varphi' X, \leq)$ are almost complete subsemilattices of $(\varphi X, \leq)$ and $\eta'_X(x) = \eta_X(x)$ for each set X and each $x \in X$.

Let $\Phi = (\varphi, \leq, \eta, \mu)$ and $\Phi' = (\varphi', \leq, \eta', \mu')$ be partially ordered monads. Φ' is a *partially ordered submonad* of Φ, if $(\varphi', \leq, \eta')$ is a basic subtriple of $(\varphi, \leq, \eta)$ and $e \circ \mu' = \mu \circ \varphi e \circ e_{\varphi'}$, where $e : \varphi' \to \varphi$ is the natural transformation consisting of all inclusion mappings $e_X : \varphi' X \to \varphi X$. Note that $\varphi e \circ e_{\varphi'} = e_\varphi \circ \varphi' e$, that is, for each set X we have $\varphi e_X \circ e_{\varphi' X} = e_{\varphi X} \circ \varphi' e_X$. Further, (φ', η', μ') is in this case also a submonad[4] of the monad (φ, η, μ).

Example 4. Let K and L be completely distributive lattices. Assume K to be a sublattice of L, with $\iota : K \to L$ being the inclusion homomorphism. Further, assume $\iota(0) = 0$ and $\iota(1) = 1$, and additionally, that $\iota(\vee_i x_i) = \vee_i \iota(x_i)$ also in the non-finite case. Define $(\iota_{id})_X : K_{id}X \to L_{id}X$ by $(\iota_{id})_X(A) = \iota \circ A$, $A : X \to K$. Then $\iota_{id} : K_{id} \to L_{id}$ becomes a natural transformation, and $\mathbf{K}_{id}$ is a submonad of $\mathbf{L}_{id}$ [4] that can be extended to being a partially ordered submonad.

[3] A set functor F' is a subfunctor of F, written $F' \leq F$, if there exists a natural transformation $e\colon F' \to F$, called the inclusion transformation, such that $e_X\colon F'X \to FX$ are inclusion maps, i.e., $F'X \subseteq FX$. The conditions on the subfunctor imply that $Ff\,|_{F'X} = F'f$ for all mappings $f : X \to Y$. Further, $\leq$ is a partial ordering.

[4] Let $\mathbf{F} = (F, \eta, \mu)$ be a monad over SET, and consider a subfunctor F' of F, with the corresponding inclusion transformation $e : F' \to F$, together with natural transformations $\eta' : id \to F'$ and $\mu' : F'F' \to F'$ satisfying the conditions $e \circ \eta' = \eta$ and $e \circ \mu' = \mu \circ Fe \circ eF'$. Then $\mathbf{F}' = (F', \eta', \mu')$ is a monad, called the submonad of $\mathbf{F}$, written $\mathbf{F}' \preceq \mathbf{F}$. Further, $\preceq$ is a partial ordering.

Example 5. Let Ω' and Ω be operator domains with $\Omega' \subseteq \Omega$, and let $\epsilon : \Omega' \to \Omega$ be the inclusion mapping. Define $\nu_X : T_{\Omega'}X \to T_\Omega X$ by $\nu_X(x) = x$, $x \in X$, and $\nu_X((n, \omega', (t'_i)_{i \le n})) = (n, \epsilon(\omega'), (\nu_X(t'_i))_{i \le n})$ for $t'_i \in T_{\Omega'}X$. Then $\nu : T_{\Omega'} \to T_\Omega X$ is a natural transformation, in fact an inclusion, and $\mathbf{T}_{\Omega'}$ is a submonad of $\mathbf{T}_\Omega$ [4], which again can be extended to being a partially ordered submonad.

5 Powerset Kleene algebras

Let $\Phi = (\varphi, \le, \eta, \mu)$ be a partially ordered monad such that always $\emptyset \in \varphi X$. Denote by $\mathsf{0}_X$, or $\mathsf{0}$ for short, the morphism $\mathsf{0} : X \to \varphi X$ satisfying $\mathsf{0}(x) = \emptyset$ for all $x \in X$, and let $\mathsf{1} = \eta_X$. Further, for $f_1, f_2 \in Hom(X, \varphi X)$, define

$$f_1 + f_2 = f_1 \vee f_2,$$

i.e. pointwise according to $(f_1 + f_2)(x) = f_1(x) \vee f_2(x)$, and

$$f_1 \cdot f_2 = f_1 \diamond f_2$$

where $f_1 \diamond f_2 = \mu_X \circ \varphi f_2 \circ f_1$ is the composition of morphisms in the corresponding Kleisli category of Φ.

A partial order $\le$ on $Hom(X, \varphi X)$ is defined pointwise, i.e. for $f_1, f_2 \in Hom(X, \varphi X)$ we say $f_1 \le f_2$ whenever $f_1(x) \le f_2(x)$ for all $x \in X$. Note that $f_1 \le f_2$ if and only if $f_1 + f_2 = f_2$.

Definition 1. *The partially ordered monad $\Phi = (\varphi, \le, \eta, \mu)$ is said to be a* Kleene monad*, if the following conditions are fulfilled:*

$$\varphi \mathsf{0}_X = \mathsf{0}_{\varphi X} \tag{3}$$
$$\varphi f(\emptyset) = \emptyset \tag{4}$$
$$\mu_X(\emptyset) = \emptyset \tag{5}$$
$$\varphi(\vee_i f_i) = \vee_i \varphi f_i \tag{6}$$
$$\varphi f \circ (\vee_i g_i) = \vee_i (\varphi f \circ g_i) \tag{7}$$
$$\mu_X \circ (\vee_i g_i) = \vee_i (\mu_X \circ g_i) \tag{8}$$

Proposition 2. *Let $\Phi = (\varphi, \le, \eta, \mu)$ be a Kleene monad.*
Then $(Hom(X, \varphi X), +, \cdot, \mathsf{0}, \mathsf{1})$ is an idempotent semiring.

Proof. We will prove the condition $(f_1 + f_2) \cdot f_3 = f_1 \cdot f_3 + f_2 \cdot f_3$. By naturality of μ and (8) we obtain

$$\begin{aligned}
(f_1 + f_2) \cdot f_3 &= \mu_X \circ \varphi f_3 \circ (f_1 + f_2) \\
&= \varphi\varphi f_3 \circ \mu_{\varphi X} \circ (f_1 + f_2) \\
&= \varphi\varphi f_3 \circ ([\mu_{\varphi X} \circ f_1] + [\mu_{\varphi X} \circ f_2]) \\
&= f_1 \cdot f_3 + f_2 \cdot f_3.
\end{aligned}$$

Other conditions are established similarly. □

The introduction of Kleene asterates is now obvious. For mappings $f : X \to \varphi X$, define

$$f^* = \bigvee_{k=0}^{\infty} f^k$$

where $f^0 = \mathsf{1}$ and $f^{k+1} = \mu_X \circ \varphi f^k \circ f$. Suprema of mappings $g : X \to Y$ is given by $(\bigwedge g)(x) = \bigwedge g(x)$.

Theorem 1. *Let $\Phi = (\varphi, \leq, \eta, \mu)$ be a Kleene monad. Then $(Hom(X, \varphi X), +, \cdot, *, \mathsf{0}, \mathsf{1})$ is a Kleene algebra.*

Proof. We will prove the condition $1 + ff^* = f^*$. We have

$$\begin{aligned} 1 + ff^* &= \mathsf{1} \vee (\mu_X \circ \varphi f^{ast} \circ f) \\ &= \mathsf{1} \vee (\mu_X \circ \varphi \bigvee_{k=0}^{\infty} f^k \circ f) \\ &= \mathsf{1} \vee \bigvee_{k=0}^{\infty} f^k \cdot f \\ &= f^0 \vee \bigvee_{k=0}^{\infty} f^{k+1} \\ &= \bigvee_{k=0}^{\infty} f^k. \end{aligned}$$

Again, other conditions are established similarly together with using conditions for partially ordered monads and Kleene monads. □

Example 6. Our main examples, the partially ordered monads $(L_{id}, \leq, \eta^{L_{id}}, \mu^{L_{id}})$ and $(T_\Omega, \leq, \eta^{T_\Omega}, \mu^{T_\Omega})$, as well as their subconstructions as described in Examples 4 and 5, are all Kleene monads.

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