CONTRIBUTIONS TO FUZZY CONVERGENCE[1]

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1. Introduction

The paper is devoted to generalized limit spaces, Cauchy spaces and limit groups defined by means of functors Φ from SET to SEMILAT. As a main application the case of a fuzzy filter functor is investigated.

For constructing completions of generalized limit groups mainly four conditions, (P), (Pr), (D) and (G), are required. They respectively concern the existence, preservation and distributivity of Φ-products, as well as requirements related to group structures. Φ-products generalize the usual notion of filter products $\mathcal{F} \times \mathcal{G}$.

We show that under (P), (Pr) and (D) the categories of generalized limit spaces and of generalized Cauchy spaces are cartesian closed. These conditions are fulfilled in the fuzzy filter case.

The fourth condition (G) is, under some restrictions, shown to be fulfilled in the fuzzy filter case. As a consequence, with these restrictions, a completion construction of fuzzy filter limit groups can be given.

2. Functors from SET to SEMILAT

Let SET denote the category of sets and SEMILAT the category of join semilattices where the morphisms are the mappings between join semilattices which preserve suprema of pairs. Each covariant functor $\Phi : \mathsf{SET} \to \mathsf{SEMILAT}$ assigns to each set X a join semilattice $\Phi X = (\varphi X, \leq)$, given as a partially ordered, and to each mapping $f : X \to Y$ a mapping $\varphi f : (\varphi X, \leq) \to (\varphi Y, \leq)$ which preserves suprema of pairs. $\varphi : \mathsf{SET} \to \mathsf{SET}$ is the underlying set functor of Φ. Frequently Φ will be written $(\varphi, \leq)$. Concerning examples we refer to [2,3]. In this paper we will only recall one of them, namely the example of fuzzy filter functor. It will be considered in section 6.

In the following let $\Phi = (\varphi, \leq)$ be a covariant functor from SET to SEMILAT.

Let X and Y be sets. For $\mathcal{M} \in \varphi X$ and $\mathcal{N} \in \varphi Y$, the Φ-*product* of $\mathcal{M}$ and $\mathcal{N}$ is defined as the greatest element $\mathcal{L}$ of $\varphi(M \times N)$, for which $\varphi\pi_1(\mathcal{L}) = \mathcal{M}$ and $\varphi\pi_2(\mathcal{L}) = \mathcal{N}$, provided it exists. Here π_1 and π_2 are the first and second projections of $X \times Y$.

In this paper some conditions will play important roles. A basic condition is the following:

(P) For all sets X, Y and all $\mathcal{M} \in \varphi X, \mathcal{N} \in \varphi Y$, the Φ-product of $\mathcal{M}$ and $\mathcal{N}$ exists.

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Under the assumption of (P), the following two conditions[4] are meaningful:

(Pr) If $f : X \to U$ and $g : Y \to V$ are any mappings and $\mathcal{M} \in \varphi X$ and $\mathcal{N} \in \varphi Y$, then $\varphi(f \times g)(\mathcal{M} \times \mathcal{N}) = \varphi f(\mathcal{M}) \times \varphi f(\mathcal{N})$.

(D) If X and Y are sets, and $\mathcal{M}_1, \mathcal{M}_2 \in \varphi X, \mathcal{N} \in \varphi Y$, then $(\mathcal{M}_1 \vee \mathcal{M}_2) \times \mathcal{N} = (\mathcal{M}_1 \times \mathcal{N}) \vee (\mathcal{M}_2 \times \mathcal{N})$.

3. Φ-Limit Spaces and Φ-Cauchy Spaces

Let Φ be a covariant functor from SET to SEMILAT, where φ is the underlying set functor of Φ. Assume that φ is *connected*, i.e. $\varphi 1$ is a singleton, say $\{\mathcal{A}\}$. Hence there is a unique natural transformation $\eta : \mathrm{id} \to \varphi$ of the identity set functor id to φ (see [2]).

Let X be a set. A φ-convergence structure t on X, i. e. a subset t of $\varphi X \times X$, is called a Φ-*limit structure* if the following conditions are satisfied, where $\mathcal{M} \xrightarrow[t]{} x$ means $(\mathcal{M}, x) \in t$.

(L1) $\eta_X(x) \xrightarrow[t]{} x$ for all $x \in X$.

(L2) $\mathcal{M} \xrightarrow[t]{} x$ and $\mathcal{N} \leq \mathcal{M}$ imply $\mathcal{N} \xrightarrow[t]{} x$.

(L3) If $\mathcal{M} \xrightarrow[t]{} x$ and $\mathcal{N} \xrightarrow[t]{} x$, then $\mathcal{M} \vee \mathcal{N} \xrightarrow[t]{} x$.

X equipped with a Φ-limit structure on X is called a Φ-*limit space*.

A mapping $f : (X, t) \to (Y, u)$ between Φ-limit spaces is said to be *continuous*, if $\mathcal{M} \xrightarrow[t]{} x$ implies $\varphi f(\mathcal{M}) \xrightarrow[u]{} f(x)$.

Let Φ-LIM denote the category of all Φ-limit spaces with all continuous mappings between these spaces as morphisms.

A Φ-*Cauchy structure* on a set X is a subset s of φX satisfying the following conditions:

(C1) $\eta_X(x) \in s$ for all $x \in X$.

(C2) $\mathcal{M} \in s$ and $\mathcal{N} \leq \mathcal{M}$ imply $\mathcal{N} \in s$.

(C3) If $\mathcal{M}, \mathcal{N} \in s$ and $\{\mathcal{M}, \mathcal{N}\}$ has a lower bound in $(\varphi X, \leq)$, then $\mathcal{M} \vee \mathcal{N} \in s$.

The pair (X, s) is called a Φ-*Cauchy space*.

Let (X, s) be a Φ-Cauchy space. The *associated* Φ-*limit structure* t consists of all pairs $(\mathcal{M}, x) \in \varphi X \times X$ such that $\mathcal{M} \vee \eta_X(x) \in s$. t is indeed a Φ-limit structure on X as has been shown in [2]. s is said to be *complete* if each Φ-Cauchy object of (X, s) converges, with respect to the associated Φ-limit structure, to some element of X.

A mapping $f : (X, s) \to (Y, u)$ between Φ-Cauchy spaces is called *Cauchy continuous* if from $\mathcal{M} \in s$ it follows $\phi f(\mathcal{M}) \in u$. Cauchy continuity implies continuity with respect to the associated Φ-limit structures, as has been shown in [2].

Let Φ-CHY denote the category of all Φ-Cauchy spaces with all Cauchy continuous mappings between these spaces as morphisms.

[4] For these conditions in a more general case using lower Φ-products, see [5]. Note that (Pr) is denoted (L) in [5].

4. Continuous Convergence and the Corresponding Cauchy Structures

In the following assume that the conditions (P), (Pr) and (D) are fulfilled. Let at first $A = (X, s)$ and $B = (Y, t)$ be Φ-limit spaces and let Z denote the set of all continuous mappings from A to B. Moreover, let $\mathrm{ev} : Z \times X \to Y$ be the evaluation mapping $(f, x) \mapsto f(x)$ $(f \in Z,\ x \in X)$.

The φ-convergence structure c on Z, defined by

$$\mathcal{K} \underset{c}{\to} f \Leftrightarrow \text{ for each } x \in X \text{ and } \mathcal{M} \underset{s}{\to} x \text{ we have } \varphi\,\mathrm{ev}(\mathcal{K} \times \mathcal{M}) \underset{t}{\to} f(x),$$

is called the *continuous convergence* on Z.

Proposition 1 *c is a Φ-limit structure.*

P r o o f . For each $f \in Z$ we have $f = \mathrm{ev} \circ (g \times 1_X) \circ \iota$ where $\iota : X \to 1 \times X$ and $g : 1 \to X$ are the mappings $x \mapsto (0, x)$ and $0 \mapsto f$, respectively. Noting that $\eta_Z(f) = \varphi g(\mathcal{A})$, where $\mathcal{A}$ is the only element of $\varphi 1$, from condition (Pr) we get $\varphi\,\mathrm{ev}(\eta_Z(f) \times \mathcal{M}) = \varphi f(\mathcal{M})$ for all $\mathcal{M} \in \varphi X$ and therefore $\eta_Z(f) \underset{c}{\to} f$. Hence (L1) is fulfilled.

Condition (L2) is obvious, and condition (L3) follows easily by means of condition (D) taking into account that Φ is ranging in SEMILAT. □

As in the filter case, continuous convergence is the coarsest Φ-limit structure on Z with respect to which ev is continuous.

There is an analogous situation for Φ-Cauchy structures, which will be considered in the following (see [1] for the filter case). Explicitely, let now $A = (X, s)$ and $B = (Y, t)$ be Φ-Cauchy spaces and let Z denote the set of all Cauchy continuous mappings from A to B. Let $\mathrm{ev} : Z \times X \to Y$ be defined as above. Moreover, let d be the subset of φZ defined as follows:

$$\mathcal{K} \in d \Leftrightarrow \text{ for each } \mathcal{M} \in s \text{ we have } \varphi\,\mathrm{ev}(\mathcal{K} \times \mathcal{M}) \in t.$$

Proposition 2 *d is a Φ-Cauchy structure.*

P r o o f . Analogous to that of Proposition 1. □

5. Cartesian Closedness of Φ-LIM and Φ-CHY

Let $A = (X, s)$, $B = (Y, t)$ and $C = (U, r)$ be Φ-limit spaces, and let $f : C \times A \to B$ be a continuous mapping.

Lemma 1. *For each $u \in U$, $f^*(u) : x \mapsto f(u, x)$ $(x \in X)$ is a continuous mapping of A into B.*

P r o o f . We have $f^*(u) = f \circ (g \times 1_X) \circ \iota$ where $\iota : X \to 1 \times X$ and $g : 1 \to U$ are the mappings $x \mapsto (0, x)$ and $0 \mapsto f$, respectively. Proving analogously as in the first part of the proof of Proposition 1, we obtain that $\varphi f(\eta_U(u) \times \mathcal{M}) = \varphi f^*(u)(\mathcal{M})$ for all $\mathcal{M} \in \varphi X$. Hence $\mathcal{M} \underset{s}{\to} x$ implies $\varphi f^*(u)(\mathcal{M}) \underset{t}{\to} f(u, x)$ and therefore $f^*(u) \in Z$. □

Clearly, $f^* : u \mapsto f^*(u)$ is the only mapping of U into Z such that

$$\mathrm{ev} \circ (f^* \times 1_X) = f. \tag{1}$$

Proposition 3 Φ-LIM *is cartesian closed.*

P r o o f . It remains to show that for each continuous mapping $f : C \times A \to B$, $f^* : C \to (Z, c)$ is continuous. Fix $\mathcal{U} \underset{r}{\to} u$. Because of the continuity of f, (1) and condition (Pr) we have $\varphi\,\mathrm{ev}(\varphi^*(\mathcal{U}) \times \mathcal{M}) = \varphi f(\mathcal{U} \times \mathcal{M}) \underset{u}{\to} f^*(u)(x)$ for all $\mathcal{M} \underset{s}{\to} x$ and therefore $\varphi f^*(\mathcal{U}) \underset{c}{\to} f^*(u)$. Hence f^* is continuous. $\square$

Proposition 4 Φ-CHY *is cartesian closed.*

P r o o f . Analogous to that of Proposition 3. $\square$

6. The Fuzzy Filter Case

Let L be a completely distributive complete lattice. A mapping $\mathcal{M} : L^X \to L$ is an *L-fuzzy filter* on X, if the following conditions are fulfilled:

(1) $\mathcal{M}(\bar{\alpha}) = \alpha$ for all $\alpha \in L$, where $\bar{\alpha} : X \to L$ is the constant mapping with value α,

(2) $\mathcal{M}(f \wedge g) = \mathcal{M}(f) \wedge \mathcal{M}(g)$ for all $f, g \in L^X$.

$f \leq g$ and $f \wedge g$ are defined componentwise.

The fuzzy filter functor $\mathsf{F}_L : \mathsf{SET} \to \mathsf{SET}$ assigns to each set X the set $\mathsf{F}_L X$ of all L-fuzzy filters on X and to each mapping $f : X \to Y$ the mapping $\mathsf{F}_L f : \mathsf{F}_L X \to \mathsf{F}_L Y$ defined for each $\mathcal{M} \in \mathsf{F}_L X$ and $g \in L^Y$ by $\mathsf{F}_L f(\mathcal{M})(g) = \mathcal{M}(g \circ f)$.

Endowing each set $\mathsf{F}_L X$ with the inversion $\leq\,\geq$ of the partial ordering $\dot{\leq}$ defined by $\mathcal{M} \dot{\leq} \mathcal{N} \iff \mathcal{M}(f) \leq \mathcal{N}(f)$ for all $f \in L^X$ gives a covariant functor

$\Phi : \mathsf{SET} \to \mathsf{SEMILAT}$, written $(\mathsf{F}_L, \dot{\geq})$ (see [3]).

Analogously to the notion of filter base there is a suitable notion of fuzzy filter base.

Let X be a set. By a *fuzzy filter base* on X we mean a non-empty subset $\mathcal{B}$ of L^X such that the following conditions are fulfilled, where for each $f \in L^X$, $\sup f$ means $\sup\{f(x) \mid x \in X\}$:

(1) $\bar{\alpha} \in \mathcal{B}$ for each $\alpha \in L$.

(2) For all $f, g \in \mathcal{B}$ there is a mapping $h \in \mathcal{B}$ such that $h \leq f \wedge g$ holds and $\sup h = \sup f \wedge \sup g$.

Each fuzzy filter base $\mathcal{B}$ on X generates a fuzzy filter $\mathcal{M}$ on X by

$$\mathcal{M}(f) = \bigvee_{g \leq f,\ g \in \mathcal{B}} \sup g. \tag{2}$$

On the other hand we have the following

Proposition 5 (see [4]) *Each fuzzy filter $\mathcal{M}$ can be generated by a fuzzy filter base on X. There even exists a greatest one, written* $\mathrm{base}\,\mathcal{M}$ *and called the* large base *of $\mathcal{M}$, which is given by* $\mathrm{base}\,\mathcal{M} = \{f \in L^X \mid \mathcal{M}(f) = \sup f\}$.

Proposition 6 *For each mapping* $f : X \to Y$, $\mathcal{M} \in \mathsf{F}_L$ *and* base $\mathcal{B}$ *of* $\mathcal{M}$, $\{g^+ \mid g \in \mathcal{B}\}$ *with* $g^+ \in L^Y$, *defined by* $g^+(y) = \bigvee_{x \in f^{-1}(y)} g(x)$ *for* $y \in f[X]$ *and* $\sup g$ *otherwise, is a base of* $\mathsf{F}_L f(\mathcal{M})$.

In the fuzzy filter case condition (P) is fulfilled. We namely have the following.

Proposition 7 *For all L-fuzzy filters* $\mathcal{M}$ *and* $\mathcal{N}$ *on sets* X *and* Y, *respectively, there exists the* $(\mathsf{F}_L, \dot{\geq})$*-product* $\mathcal{M} \times \mathcal{N}$ *of* $\mathcal{M}$ *and* $\mathcal{N}$, *and* $\mathcal{B} = \{f \circ \pi_1 \wedge g \circ \pi_2 \mid f \in \mathcal{C}, g \in \mathcal{D}\}$ *is a base of* $\mathcal{M} \times \mathcal{N}$, *where* $\mathcal{C}$ *and* $\mathcal{D}$ *are bases of* $\mathcal{M}$ *and* $\mathcal{N}$, *respectively, and* π_1 *and* π_2 *are the first and second projections of* $X \times Y$.

P r o o f . Obviously, $\mathcal{B}$ is the base of some fuzzy filter $\mathcal{L}$ on $X \times Y$. Representing $\mathcal{L}$ according to (2) by means of $\mathcal{B}$, shows that $\mathcal{L} = \mathcal{M} \times \mathcal{N}$. □

For each set X and $\mathcal{M} \in \varphi X$ let $d_X(\mathcal{M}) = \{U \subseteq X \mid \varphi\iota_U(\mathcal{N}) = \mathcal{M}$ for some $\mathcal{N} \in \varphi U\}$, where ι_U is the inclusion mapping of U into X.

Lemma 2. $M \in d_X(\mathcal{M})$ *holds if and only if for all* $f, g \in L^X$ *with* $f|M = g|M$ *we have* $\mathcal{M}(f) = \mathcal{M}(g)$.

Lemma 3. *From* $M \in d_X(\mathcal{M})$ *and* $N \in d_X(\mathcal{N})$ *it follows* $M \times N \in d_{X \times Y}(\mathcal{M} \times \mathcal{N})$.

Proposition 8 $(\mathsf{F}_L, \dot{\geq})$ *fulfills condition* (Pr).

P r o o f . Follows by means of Propositions 6 and 7, Lemma 2 and Lemma 3. □

Lemma 4. *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be L-fuzzy filters on a set* X. *Then* $\text{base}\,(\mathcal{M} \vee \mathcal{N}) = \text{base}\,\mathcal{M} \cap \text{base}\,\mathcal{N}$ *and* $f \in \text{base}\,\mathcal{M}$ *and* $g \in \text{base}\,\mathcal{N}$ *imply* $f \wedge g \in \text{base}\,(\mathcal{M} \vee \mathcal{N})$ *and* $\sup\,(f \wedge g) = \sup f \wedge \sup g$.

Proposition 9 $(\mathsf{F}_L, \dot{\geq})$ *also fulfills condition* (D).

P r o o f . Let $\mathcal{M}_1, \mathcal{M}_2 \in \mathsf{F}_L X$ and $\mathcal{N} \in \mathsf{F}_L Y$. Representing $\mathcal{M}_1 \times \mathcal{N}$, $\mathcal{M}_2 \times \mathcal{N}$ and $(\mathcal{M}_1 \vee \mathcal{M}_2) \times \mathcal{N}$ according to (2) by means of related bases as given in Proposition 7 and taking into account Lemma 4 and that L is completely distributive, shows that $(\mathcal{M}_1 \vee \mathcal{M}_2) \times \mathcal{N} = (\mathcal{M}_1 \times \mathcal{N}) \vee (\mathcal{M}_2 \times \mathcal{N})$. □.

7. Completions of Φ-Limit Groups

The papers [2] and [4], respectively, contain the Wyler and Kowalsky completions of Φ-Cauchy spaces. Whereas in [4] the fuzzy filter case is included, for the Wyler completion of Φ-Cauchy spaces, the fuzzy filter case is considered separately in [3].

In [5] the Wyler completion of Φ-limit groups is presented according to the construction given by G. Kneis in [6]. Clearly, a Φ-limit group is a Φ-limit space equipped with a group structure such that the group operations are continuous mappings.

For this type of completion the conditions (P), (Pr) and (D) are needed. Moreover, a further condition (G) is used, which, under the assumption that (P) is fulfilled, can be formulated as follows:

(G) For each group (X, g) and $\mathcal{M} \in \varphi X$, whenever there is a subset U of X and a $\mathcal{G} \in \varphi U$ with $\varphi\nu(\mathcal{M} \times \mathcal{M}) \leq \varphi\iota_U(\mathcal{G})$, then there exists a finite subset E of X and an $\mathcal{N} \in \varphi V$ with $\mathcal{M} \leq \varphi\iota_V(\mathcal{N})$. Here $\nu : X \times X \to X$ denotes the mapping $(x, y) \mapsto x \cdot y^{-1}$ and V the algebraic closure of $U \cup E$, and ι_U and ι_V are the inclusion mappings of U and V into X, respectively.

Proposition 10 *Under the assumption*

($*$) *L is a complete chain and each $\alpha \in L,\ \alpha > 0$ has a predecessor,*

condition (G) *is fulfilled for the fuzzy filter functor* $(\mathsf{F}_L, \dot{\geq})$.

P r o o f . Let (X, g), $\mathcal{M}$, U and $\mathcal{G}$ be given as in condition (G). Further let $i : X \to L$ be the mapping which has the value 1 over U and the value 0 over $X \setminus U$. Then $\varphi\nu(\mathcal{M} \times \mathcal{M})(i) = (\mathcal{M} \times \mathcal{M})(i \circ \nu) = 1$ and therefore

$$1 = \bigvee_{\substack{f,g \in \mathrm{base}\,\mathcal{M} \\ f\circ\pi_1 \wedge g\circ\pi_2 \leq i\circ\nu}} \sup f \wedge \sup g = \bigvee_{\substack{h \in \mathrm{base}\,\mathcal{M} \\ x\cdot y^{-1} \notin U \Rightarrow h(x)=0 \text{ or } h(x)=0}} \sup h \ .$$

Because of ($*$), hence there exist $h \in \mathrm{base}\,\mathcal{M}$ and $x_0 \in X$ such that $h(x_0) = 1$ and $h(x) = 0$ for all $x \in X \setminus U \cdot x_0$. Then $h(x) = 0$ for all $x \in X \setminus V$, where V is the algebraic closure of $U \cup \{x_0\}$. Now $\{g \wedge h \mid g \in \mathrm{base}\,\mathcal{M}\} \cup \{\bar{\alpha} \mid \alpha \in L\}$ is a fuzzy filter base, and because of $\mathcal{M}(f) = \bigvee_{\substack{g\wedge h \leq f \\ g \in \mathrm{base}\,\mathcal{M}}} \sup\,(g \wedge h)$ for all $f \in L^X$, it is even a base of $\mathcal{M}$. Thus, $\mathcal{M}(f)$ only depends on $f\,|\,U$ for all $f \in L^X$. But this implies that there exists an $\mathcal{N} \in \varphi V$ such that $\mathcal{M} = \varphi\iota_V(\mathcal{N})$. □

As a consequence of this proposition we get that in the fuzzy filter case , under the assumption ($*$), the Wyler completion of Φ-limit spaces can be constructed. For more details on this completion we refer to [5]. Note that the condition ($*$) already is used in [4] for constructing the Richardson compactification in the fuzzy filter case.

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