Random Context Picture Grammars: The State of the Art

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Abstract. We present a summary of results on random context picture grammars (rcpgs), which are a method of syntactic picture generation. The productions of such a grammar are context-free, but their application is regulated—permitted or forbidden—by context randomly distributed in the developing picture. Thus far we have investigated three important subclasses of rcpgs, namely random permitting context picture grammars, random forbidding context picture grammars and table-driven context-free picture grammars. For each subclass we have proven characterization theorems and shown that it is properly contained in the class of rcpgs. We have also developed a characterization theorem for all picture sets generated by rcpgs, and used it to find a set that cannot be generated by any rcpg.

Key words: formal languages, picture grammars, syntactic picture generation, image analysis, random context grammars, scene understanding

1 Introduction

Picture generation is a challenging task in Computer Science and applied areas, such as document processing (character recognition), industrial automation (inspection) and medicine (radiology).

Syntactic methods of picture generation have become established during the last decade or two. A variety of methods is discussed and extensive lists of references are given in [9, 11, 12]. Random context picture grammars (rcpgs) [7] generate pictures through successive refinement. They are context-free grammars with regulated rewriting; the motivation for their development was the fact that context-free grammars are often too weak to describe a given picture set, eg. the approximations of the Sierpiński carpet, while context-sensitive grammars are too complex to use.

Random context picture grammars have at least three interesting subclasses, namely random permitting context picture grammars (rPcpgs), random forbidding context picture grammars (rFcpgs) and table-driven context-free picture grammars (Tcfpgs). For each of these classes we have developed characterization theorems. In particular, for rPcpgs we proved a pumping lemma and used it to show that these grammars are strictly weaker than rcpgs [5]. For rFcpgs we proved a shrinking lemma [4], and showed that they too are strictly weaker than rcpgs [6]. In the case of Tcfpgs, we developed two characterization theorems and showed that these grammars are strictly weaker than rFcpgs [1].

Finally, we have developed a characterization theorem for all galleries generated by rcpgs, and used it to find a picture set, more commonly known as a gallery, that cannot be generated by any rcpg [13].

F. Drewes, A. Habel, B. Hoffmann, D. Plump (Eds.): Manipulation of Graphs, Algebras and Pictures. Essays Dedicated to Hans-Jörg Kreowski on the Occasion of His 60th Birthday, pp. 135–147, 2009.

In this paper we present a summary of the above results. We formally define rcpgs in Section 2. In Section 3 we present the pumping lemma for rPcpgs, and use it to show that no rPcpg can generate the approximations of the Sierpiński carpet. In Section 4 we state the shrinking lemma for rFcpgs, and present a gallery that cannot be generated by any rFcpg. Then, in Section 5, we define Tcfpgs, present two characterization theorems for these grammars and show that they are strictly weaker than rFcpgs. In Section 6 we present a property of all galleries generated with rcpgs, and then construct a gallery that does not belong to this class. We briefly touch on a generalization of rcpgs in Section 7. Future work is recommended in Section 8.

2 Random context picture grammars

In this section we introduce random context picture grammars. For picture grammars we need a geometric context; we choose the situation of squares divided into equal squares.

In the following, let $\mathbb{N}_+ = \{1, 2, 3, \ldots\}$. For $k \in \mathbb{N}_+$, let $[k] = \{1, 2, \ldots, k\}$.

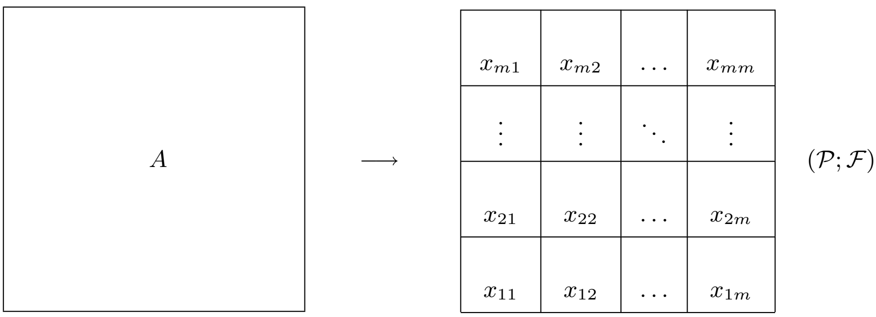


Fig. 1. Production.

Random context picture grammars generate pictures using productions of the form in Figure 1, where A is a variable, $m \in \mathbb{N}_+$, $x_{12}, \ldots, x_{mm}$ are variables or terminals, and $\mathcal{P}$ and $\mathcal{F}$ are sets of variables. The interpretation is as follows: if a developing picture contains a square labelled A and if all variables of $\mathcal{P}$ and none of $\mathcal{F}$ appear as labels of squares in the picture, then the square labelled A may be divided into equal squares with labels $x_{11}, x_{12}, \ldots, x_{mm}$.

We denote a square by a lowercase Greek letter, eg., (A, α) denotes a square α labelled A. If α is a square, $\alpha_{11}, \alpha_{12}, \ldots, \alpha_{mm}$ denote the equal subsquares into which α can be divided, with, eg., α_{11} denoting the left bottom one.

A *random context picture grammar* $G = (V_\mathrm{N}, V_\mathrm{T}, P, (S, \sigma))$ has a finite alphabet V of *labels*, consisting of disjoint subsets V_N of *variables* and V_T of *terminals*. P is a finite set of *productions* of the form $A \to [x_{11}, x_{12}, \ldots, x_{mm}](\mathcal{P}; \mathcal{F})$,

$m \in \mathbb{N}_+$, where $A \in V_\mathrm{N}$, $x_{11}, x_{12}, \ldots, x_{mm} \in V$ and $\mathcal{P}, \mathcal{F} \subseteq V_\mathrm{N}$. Finally, there is an *initial labelled square* (S, σ) with $S \in V_\mathrm{N}$.

A *pictorial form* is any finite set of nonoverlapping labelled squares in the plane. If Π is a pictorial form, we denote by $l(\Pi)$ the set of labels used in Π.

Thirdly, the *size* of a pictorial form Π is the number of squares contained in it, denoted $|\Pi|$.

For an rcpg G and pictorial forms Π and Γ we write $\Pi \Longrightarrow_\mathrm{G} \Gamma$ if there is a production $A \to [x_{11}, x_{12}, \ldots, x_{mm}](\mathcal{P}; \mathcal{F})$ in G, Π contains a labelled square (A, α), $l(\Pi \backslash \{(A, \alpha)\}) \supseteq \mathcal{P}$ and $l(\Pi \backslash \{(A, \alpha)\}) \cap \mathcal{F} = \emptyset$, and $\Gamma = (\Pi \backslash \{(A, \alpha)\}) \cup \{(x_{11}, \alpha_{11}), (x_{12}, \alpha_{12}), \ldots, (x_{mm}, \alpha_{mm})\}$. As usual, $\Longrightarrow_\mathrm{G}^*$ denotes the reflexive transitive closure of $\Longrightarrow_\mathrm{G}$.

If every production in G has $\mathcal{P} = \mathcal{F} = \emptyset$, we call G a *context-free picture grammar (cfpg)*; if $\mathcal{F} = \emptyset$ for every production, G is a *random permitting context picture grammar*, and when $\mathcal{P} = \emptyset$, G is a *random forbidding context picture grammar*. The *gallery* $\mathcal{G}(G)$ *generated by a grammar* $G = (V_\mathrm{N}, V_\mathrm{T}, P, (S, \sigma))$ is $\{\Phi \mid \{(S, \sigma)\} \Longrightarrow_\mathrm{G}^* \Phi \text{ and } l(\Phi) \subseteq V_\mathrm{T}\}$. An element of $\mathcal{G}(G)$ is called a *picture*.

Let Φ be a picture in the square σ. For any $m \in \mathbb{N}_+$, let σ be divided into equal subsquares, say $\sigma_{11}, \sigma_{12}, \ldots, \sigma_{mm}$. A *subpicture* Γ of Φ is any subset of Φ that fills a square $\sigma_{ij}, i, j \in [m]$, i.e., the union of all the squares in Γ is the square σ_{ij}.

Finally, please note that we write a production $A \to [x_{11}](\mathcal{P}; \mathcal{F})$ as $A \to x_{11}(\mathcal{P}; \mathcal{F})$.

3 Permitting context only

In this section we concentrate on grammars that use permitting context only. We present a pumping lemma for the corresponding galleries, and show that rPcpgs cannot generate $\mathcal{G}_\mathrm{carpet}$, the gallery of approximations of the Sierpiński carpet.

We first introduce some notation. Let Π be a pictorial form that occupies a square α, i.e., the union of all the squares in Π is the square α; this we denote by (Π, α). Let β be any square in the plane. Then $(\Pi \to \beta)$ denotes the pictorial form obtained from Π by uniformly scaling (up or down) and translating all the labeled squares in Π to fill the square β, retaining all the labels.

The pumping lemma for rPcpgs and some corollaries are proven in [5]. It states:

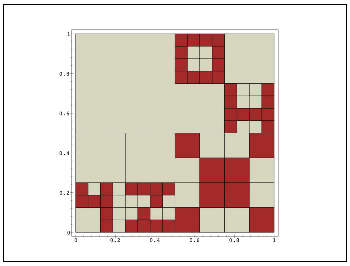
Theorem 1. *For any rPcpg G there is an $m \in \mathbb{N}_+$ such that for any picture $\Phi \in \mathcal{G}(G)$ with $|\Phi| \geq m$ there is a number l, $l \in [m]$, such that:*

1. *Φ contains l mutually disjoint nonempty subpictures $(\Omega_1, \alpha_1), \ldots, (\Omega_l, \alpha_l)$ and l mutually disjoint nonempty subpictures $(\Psi_1, \beta_1), \ldots, (\Psi_l, \beta_l)$, these being related by a function $\vartheta : \{1, \ldots, l\} \to \{1, \ldots, l\}$ such that for each i, $i \in [l]$, $\beta_i \subseteq \alpha_{\vartheta(i)}$ and for at least one i, $i \in [l]$, $\beta_i \subsetneqq \alpha_{\vartheta(i)}$;*
2. *the picture obtained from Φ by substituting $(\Omega_i \to \beta_i)$ for (Ψ_i, β_i) for all i, $i \in [l]$, is in $\mathcal{G}(G)$;*

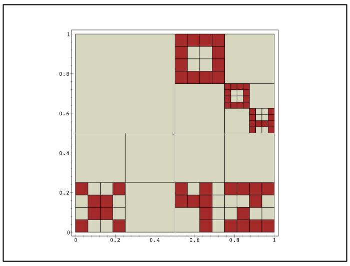
3. *recursively carrying out the operation described in (2) always results in a picture in $\mathcal{G}(G)$.*

Example 1. Consider Φ^1 in Figure 2(a). Let (Ω_1, α_1) be the lower left hand quarter, (Ω_2, α_2) the lower right hand quarter, (Ω_3, α_3) the upper left hand quarter and (Ω_4, α_4) the upper right hand quarter of Φ^1. Furthermore, let (Ψ_1, β_1) be equal to (Ω_2, α_2), (Ψ_2, β_2) the letter Y, (Ψ_3, β_3) the letter Z and (Ψ_4, β_4) the letter H. Then $\vartheta(1) = 2$, $\vartheta(2) = \vartheta(3) = 1$ and $\vartheta(4) = 4$.

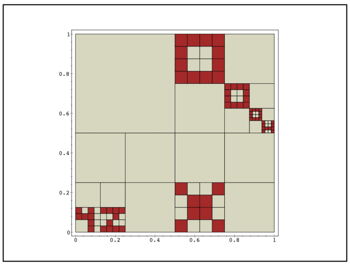
We obtain Φ^2 in Figure 2(b) by substituting $(\Omega_i \rightarrow \beta_i)$ for (Ψ_i, β_i), $i \in [4]$, in Φ^1. Then we obtain Φ^3 in Figure 2(c) by carrying out this operation on Φ^2.



(a) Φ^1



(b) Φ^2



(c) Φ^3

Fig. 2. Pumping Φ^1.

An immediate consequence of the pumping property is that the set of sizes of the pictures in an infinite gallery generated by an rPcpg contains an infinite arithmetic progression. From this it follows that $\mathcal{G}_{\text{carpet}}$, two pictures of which are shown in Figure 3, cannot be generated using permitting context only. This gallery can be created by an rFcpg, as is shown in [1].

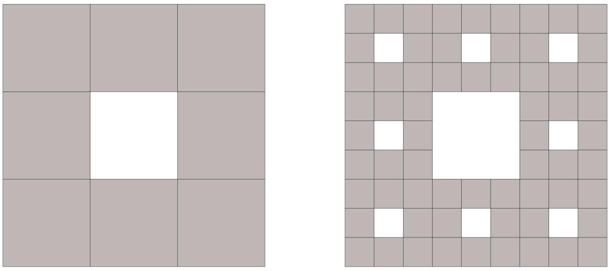


Fig. 3. Two pictures from $\mathcal{G}_{\mathrm{carpet}}$.

4 Forbidding context only

In this section we concentrate on grammars that use forbidding context only and present a shrinking lemma for the corresponding galleries. The lemma is proven in [4] and states:

Theorem 2. *Let G be an rFcpg. For any integer $t \geq 2$ there exists an integer $k = k(t)$ such that for any picture $\Phi \in \mathcal{G}(G)$ with $|\Phi| \geq k$ there are t pictures $\Phi^1, \ldots, \Phi^t = \Phi$ in $\mathcal{G}(G)$ and $t-1$ numbers $l_2, \ldots, l_t$ such that for each j, $2 \leq j \leq t$,*

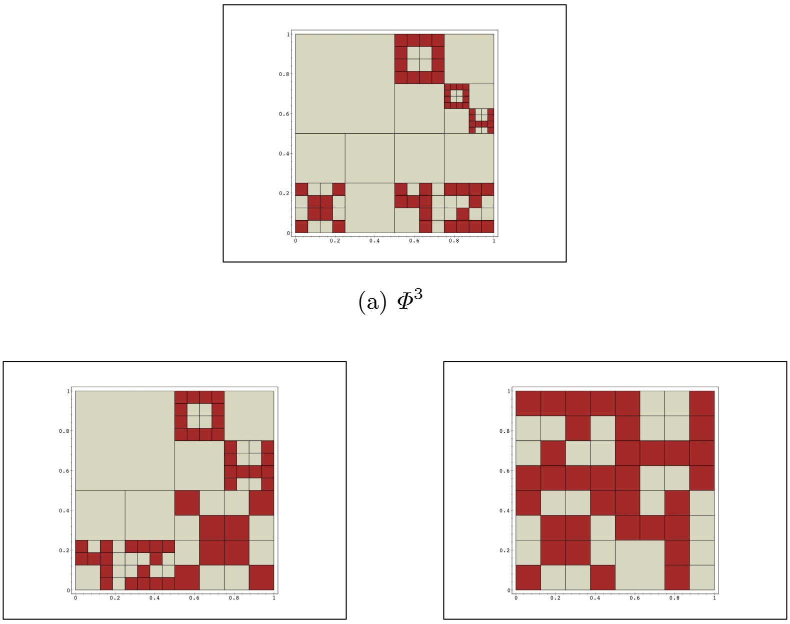
1. *Φ^j contains l_j mutually disjoint nonempty subpictures (Φ_{j1}, α_{j1}), …, $(\Phi_{jl_j}, \alpha_{jl_j})$ and l_j mutually disjoint nonempty subpictures (ϕ_{j1}, β_{j1}), …, $(\phi_{jl_j}, \beta_{jl_j})$, these being related by a function $\vartheta_j : \{1, \ldots, l_j\} \to \{1, \ldots, l_j\}$ such that for each i, $i \in [l_j]$, $\beta_{ji} \subseteq \alpha_{j\vartheta_j(i)}$ and for at least one i, $i \in [l_j]$, $\beta_{ji} \subsetneq \alpha_{j\vartheta_j(i)}$;*
2. *the picture Φ^{j-1} is obtained by substituting $(\phi_{ji} \to \alpha_{ji})$ for (Φ_{ji}, α_{ji}) for all i, $i \in [l_j]$, in Φ^j.*

Example 2. Consider Φ^3 in Figure 4(a). We can choose (Φ_{31}, α_{31}) as the lower left hand quarter, (Φ_{32}, α_{32}) as the lower right hand quarter and (Φ_{33}, α_{33}) as the upper right hand quarter of Φ^3, furthermore (ϕ_{31}, β_{31}) equal to (Φ_{32}, α_{32}), (ϕ_{32}, β_{32}) as the letter X and (ϕ_{33}, β_{33}) as the lower right hand quarter of (Φ_{33}, α_{33}). Here $l_3 = 3$ and $\vartheta_3(1) = 2$, $\vartheta_3(2) = 1$ and $\vartheta_3(3) = 3$.

Φ^2 in Figure 4(b) is obtained by substituting $(\phi_{3i} \to \alpha_{3i})$ for (Φ_{3i}, α_{3i}), $1 \leq i \leq 3$, in Φ^3.

Now consider Φ^2 in Figure 4(b). We can choose (Φ_{21}, α_{21}) as the lower left hand quarter, (Φ_{22}, α_{22}) as the lower right hand quarter, (Φ_{23}, α_{23}) as the upper left hand quarter and (Φ_{24}, α_{24}) as the upper right hand quarter of Φ^2, furthermore (ϕ_{21}, β_{21}) equal to (Φ_{22}, α_{22}), (ϕ_{22}, β_{22}) as the letter Y, (ϕ_{23}, β_{23}) as the letter Z and (ϕ_{24}, β_{24}) as the letter H. Here $l_2 = 4$ and $\vartheta_2(1) = 2$, $\vartheta_2(2) = \vartheta_2(3) = 1$ and $\vartheta_2(4) = 4$.

Φ^1 in Figure 4(c) is obtained by substituting $(\phi_{2i} \to \alpha_{2i})$ for (Φ_{2i}, α_{2i}), $1 \leq i \leq 4$, in Φ^2.



(a) Φ^3

(b) Φ^2

(c) Φ^1

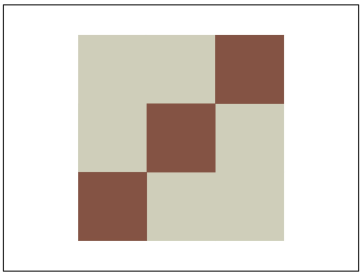
Fig. 4. Shrinking Φ^3.

In [6] we use the technique developed for the proof of the shrinking lemma to show that a certain gallery, $\mathcal{G}_{\text{trail}}$, cannot be generated by any rFcpg, but can be generated by an rcpg. Therefore rFcpgs are strictly weaker than rcpgs.

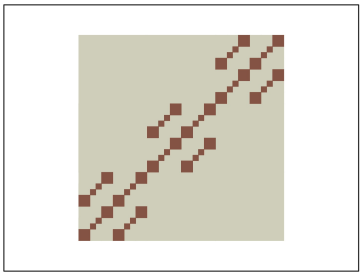
Consider $\mathcal{G}_{\text{trail}} = \{\Phi^1, \Phi^2, \ldots\}$, where Φ^1, Φ^2 and Φ^3 are shown in Figures 5(a), 5(b) and 5(c), respectively. For the sake of clarity, an enlargement of the bottom left hand ninth of Φ^3 is given in Figure 5(d).

For $i = 2, 3, \ldots$, Φ^i is obtained by dividing each dark square in Φ^{i-1} into four and placing a copy of Φ^1, modified so that it has exactly $i + 2$ dark squares, all on the bottom left to top right diagonal, into each quarter.

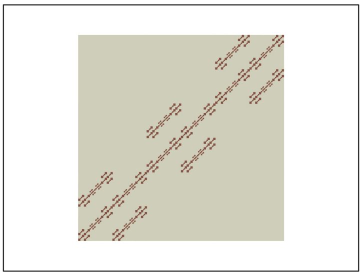
The modification of Φ^1 is effected in its middle dark square only and proceeds in detail as follows: The square is divided into four and the newly-created bottom left hand quarter coloured dark. The newly-created top right hand quarter is again divided into four and its bottom left hand quarter coloured dark. This successive quartering of the top right hand square is repeated until a total of $i - 1$ dark squares have been created, then the top right hand square is also coloured dark. The new dark squares thus get successively smaller, except for the last two, which are of equal size.



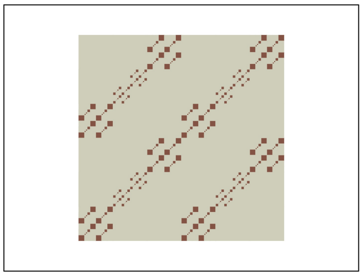
(a) Φ^1



(b) Φ^2



(c) Φ^3



(d) Bottom left hand ninth of Φ^3 enlarged

Fig. 5. The pictures Φ^1, Φ^2 and Φ^3 from $\mathcal{G}_{\mathrm{trail}}$.

5 Table-driven context-free picture grammars

In [1] we introduce table-driven context-free picture grammars, and compare them to cfpgs, rPcpgs and rFcpgs. We also give two necessary conditions for a gallery to be generated by a Tcfpg, and use them to find galleries that cannot be made by any Tcfpg.

A *table-driven context-free picture grammar* is a system $G = (V_{\mathrm{N}}, V_{\mathrm{T}}, \mathcal{T}, (S, \sigma))$, where V_{N}, V_{T}, $V = V_{\mathrm{N}} \cup V_{\mathrm{T}}$ and (S, σ) are as defined in Section 2. $\mathcal{T}$ is a finite set of *tables*, each table $R \in \mathcal{T}$ satisfying the following two conditions:

1. R is a finite set of productions of the form $A \rightarrow [x_{11}, x_{12}, \ldots, x_{mm}]$, $m \in \mathbb{N}_+$, where $A \in V_{\mathrm{N}}$, and $x_{11}, x_{12}, \ldots, x_{mm} \in V$.
2. R is complete, i.e., for each $A \in V_{\mathrm{N}}$, there exist an $m \in \mathbb{N}_+$ and $x_{11}, x_{12}, \ldots, x_{mm} \in V$ such that $A \rightarrow [x_{11}, x_{12}, \ldots, x_{mm}]$ is in R.

As in the case of rcpgs, the squares containing variables are replaced, but the terminals are never rewritten. Every direct derivation must replace all variables in the pictorial form; the completeness condition ensures that this is possible.

For any production p, say $A \to [x_{11}, x_{12}, \ldots, x_{mm}]$, A is called the left hand side of p, and $[x_{11}, x_{12}, \ldots, x_{mm}]$ the right hand side of p, denoted by $\mathrm{lhs}\,(p)$ and $\mathrm{rhs}\,(p)$, respectively.

For a labelled square (A, α) and a production p with $A = \mathrm{lhs}\,(p)$, say $A \to [x_{11}, x_{12}, \ldots, x_{mm}]$, $m \in \mathbb{N}_+$, we denote $\{(x_{11}, \alpha_{11}), (x_{12}, \alpha_{12}), \ldots, (x_{mm}, \alpha_{mm})\}$ by $\mathrm{repl}\,((A, \alpha))\,p$ [1].

For pictorial form Π, we define $\mathrm{var}\,(\Pi) = \{(A, \alpha) \in \Pi \mid A \in V_{\mathrm{N}}\}$. For pictorial form Π and table R, we call $\mathrm{b} : \mathrm{var}\,(\Pi) \to R$ a *base* [2] on Π if for each $(A, \alpha) \in \mathrm{var}\,(\Pi)$, $\mathrm{lhs}\,(\mathrm{b}\,((A, \alpha))) = A$.

Let Π and Γ be pictorial forms. We say that Π directly derives Γ ($\Pi \Longrightarrow \Gamma$) if there exists a base b on Π such that

$$\Gamma = \Pi \setminus \mathrm{var}\,(\Pi) \cup \bigcup_{(A,\alpha) \in \mathrm{var}(\Pi)} \mathrm{repl}\,((A, \alpha))\,\mathrm{b}\,((A, \alpha)).$$

For Tcfpgs, the terms $\Longrightarrow_G^*$, *gallery*, and *picture* are defined as for rcpgs in Section 2.

Finally, please note that we write a production $A \to [x_{11}]$ as $A \to x_{11}$.

In [1] we present a Tcfpg that generates $\mathcal{G}_{\mathrm{carpet}}$. From this it follows that Tcfpgs can generate a gallery that no rPcpg can and that Tcfpgs are strictly more powerful than context-free picture grammars.

In [1] we state two necessary conditions for a gallery to be generated by a Tcfpg.

Before we can state the first such condition, we need a definition. Let Π be a pictorial form and B a set. Then $\#_B\,(\Pi)$ denotes the number of occurrences of elements of B in Π.

Theorem 3. *Let $\mathcal{G}$ be a gallery generated by a Tcfpg with terminal alphabet V_{T}. Then for every $B \subseteq V_{\mathrm{T}}$, $B \neq \emptyset$, there exists a positive integer k such that, for every picture $\Phi \in \mathcal{G}$ either*

1. *$\#_B\,(\Phi) \leq 1$, or*
2. *Φ contains a subpicture Ψ such that $|\Psi| \leq k$ and $\#_B\,(\Psi) \geq 2$, or*
3. *there exist infinitely many $\Upsilon \in \mathcal{G}$ such that $\#_B\,(\Upsilon) = \#_B\,(\Phi)$.*

In [1] we use Theorem 3 to show that a certain gallery, $\mathcal{G}_{\mathrm{not-Tcfpg}}$, cannot be generated by any Tcfpg. Consider $\mathcal{G}_{\mathrm{not-Tcfpg}} = \{\Phi^1, \Phi^2, \ldots\}$, where Φ^1, Φ^2 and Φ^3 are given in Figure 6 from left to right. Let the terminals b, g and w represent squares with the colours black, grey and white respectively. Then Φ^n, $n \in \mathbb{N}_+$, is such that the terminals on its diagonal, read from bottom left to top right, form the string $b\,(bg^n)^n$, while the rest of the picture is white.

Before we can state the second necessary condition for a gallery to be generated by a Tcfpg, we introduce the properties *nonfrequent* and *rare*, which are based on properties presented in [3]. Let $\mathcal{G}$ be a set of pictures with labels from the alphabet V_{T}, and B a nonempty subset of V_{T}. Then

[1] The use of "repl" was inspired by the concept *repl* defined in [2].
[2] The use of "base" was inspired by the concept *base* defined in [2].

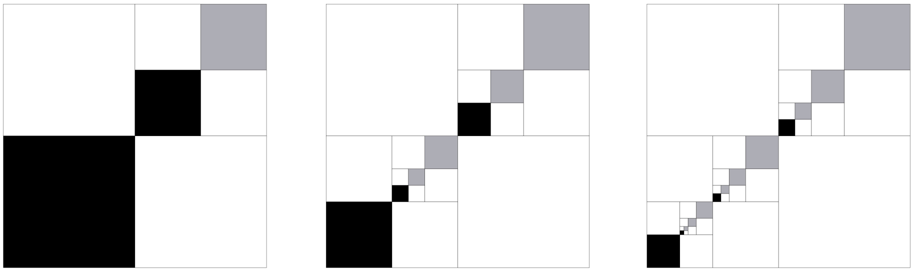


Fig. 6. The pictures Φ^1, Φ^2 and Φ^3 from the gallery $\mathcal{G}_{\mathrm{not-Tcfpg}}$.

- B is called *nonfrequent* in $\mathcal{G}$ if there exists a constant k such that for every $\Phi \in \mathcal{G}$, $\#_B(\Phi) < k$.
- B is *rare* in $\mathcal{G}$ if for every $k \in \mathbb{N}_+$ there exists an $n_k > 0$ such that for every $n \in \mathbb{N}$ with $n > n_k$, if a picture $\Phi \in \mathcal{G}$ contains n occurrences of letters from B then for each two such occurrences, the smallest subpicture containing those occurrences has size at least k.

Theorem 4. *Let $G = (V_\mathrm{N}, V_\mathrm{T}, \mathcal{T}, (S, \sigma))$ be a Tcfpg and $B \subseteq V_\mathrm{T}, B \neq \emptyset$. If B is rare in $\mathcal{G}(G)$, then B is nonfrequent in $\mathcal{G}(G)$.*

In [1] we use Theorem 4 to show that a certain gallery, $\mathcal{G}_{\mathrm{rFcpg-Tcfpg}}$, cannot be generated by any Tcfpg. Consider $\mathcal{G}_{\mathrm{rFcpg-Tcfpg}} = \{\Phi^{m,n} \mid n \geq 1, m \geq n\}$, where $\Phi^{2,2}$ and $\Phi^{4,3}$ are given in Figure 7 from left to right. Let the terminals b, g and w represent squares with the colours black, grey and white respectively. Then $\Phi^{m,n}$ is such that the terminals on its diagonal, read from bottom left to top right, form the string $(bg^m)^n$, while the rest of the picture is white.

In [1] we show that every gallery generated by a Tcfpg can be generated by an rFcpg. Then we present an rFcpg that generates the gallery $\mathcal{G}_{\mathrm{rFcpg-Tcfpg}}$. From that it follows that Tcfpgs are strictly weaker than rFcpgs.

6 The Limitations of Random Context

In [13] we investigate the limitations of random context picture grammars. First we study those grammars that generate only pictures that are composed of squares of equal size and show that the corresponding galleries enjoy a certain commutativity. This enables us to construct a set of pictures that cannot be generated by any rcpg. Then we generalize the commutativity theorem to the class of all rcpgs.

For the sake of simplicity, we consider only rcpgs of which every production that effects a subdivision produces exactly four subsquares. Also, we let σ be the unit square $((0,0),(1,1))$. The result we state below can be formulated for the case of rcpgs with productions that effect other subdivisions [13].

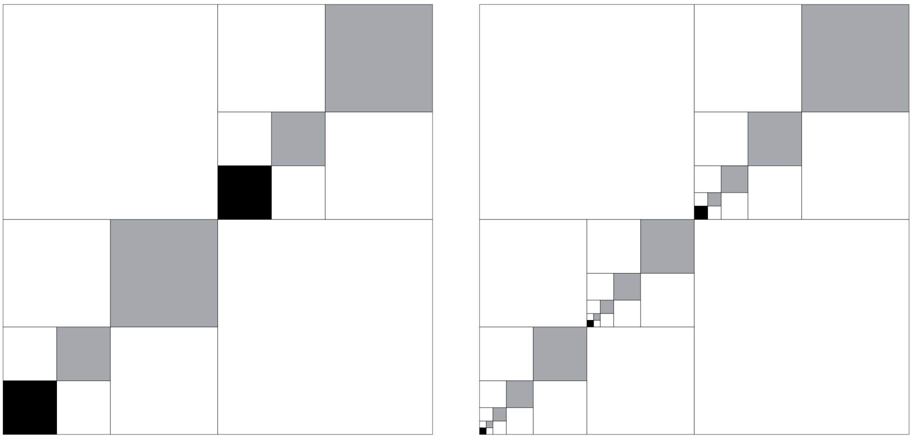


Fig. 7. The pictures $\Phi^{2,2}$ and $\Phi^{4,3}$ from the gallery $\mathcal{G}_{\mathrm{rFcpg-Tcfpg}}$.

Before we can state the theorem, we need some definitions. A picture is called *n-divided*, for $n \in \mathbb{N}_+$, if it consists of 4^n equal subsquares, each labeled with a terminal. For example, the picture on the left hand side of Figure 8 is 4-divided. A *level-m subsquare* of an n-divided picture, with $1 \leq m \leq n$, is a square $((x2^{-m}, y2^{-m}), ((x+1)2^{-m}, (y+1)2^{-m}))$, where x and y are integers and $0 \leq x, y < 2^m$. Note that, for $m < n$, a level-m subsquare consists of all 4^{n-m} labeled subsquares contained in it. For example, the upper left hand quarter of the above mentioned picture is a level-1 subsquare of the picture and consists of 4^3 labeled subsquares.

Two n-divided pictures Φ^1 and Φ^2 are said to *commute at level m* if Φ^1 contains two different level-m subsquares α and β such that Φ^2 can be obtained by simply interchanging the labeling of α and β. A picture Φ^1 is called *self-commutative at level m* if Φ^1 and Φ^1 commute at level m.

In [13] we give a proof of the following theorem:

Theorem 5. *Let $G = (V_{\mathrm{N}}, V_{\mathrm{T}}, P, (S, \sigma))$ be an rcpg that generates an infinite gallery of n-divided pictures, where $n \in \mathbb{N}_+$. Then there exist an m and a c such that each picture that is c-divided is either self-commutative at level m or commutes with another picture in the gallery at level m.*

We now use Theorem 5 to construct a gallery that cannot be generated by any rcpg.

Let $m \in \mathbb{N}_+$. Consider the $2m$-divided picture Φ that is constructed as follows: For any level-m subsquare α in Φ, if α is in row i and column j of Φ, then the level-$2m$ subsquare in row i and column j of α is coloured dark. All the other level-$2m$ subsquares are coloured light.

For example, in Figure 8, $m = 2$. The picture on the left hand side is 2×2-divided, i.e., 4-divided. On the right hand side, we show the level-2 subsquares

α_1 in row 2, column 2, and α_2 in row 3, column 4. The level-4 subsquare in row 2, column 2 of α_1 is coloured dark and all other level-4 subsquares of α_1 are coloured light. Similarly, the level-4 subsquare in row 3, column 4 of α_2 is coloured dark and all other level-4 subsquares of α_2 are coloured light.

Then Φ is not self-commutative at level m. Thus we have:

Theorem 6. *There exists a set of pictures, each consisting of the unit square subdivided into equal subsquares and coloured with two colours, that cannot be generated by an rcpg.*

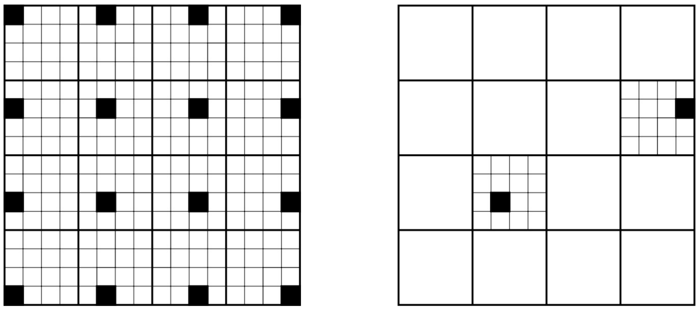


Fig. 8. 4-divided.

In [13] we generalize Theorem 5 to the class of all rcpgs.

7 Generalized Random Context Picture Grammars

As geometric context for random context picture grammars we used squares divided into equal, non-overlapping squares. Clearly we could start with another shape, eg. a triangle, and divide it successively into non-overlapping triangles of equal size. We could also divide a shape into shapes that do not all have the same size or have the same shape as the original. For any given gallery there may be a combination of shapes and arrangement of subshapes that is most effective. This leads us to a generalization of rcpgs, so-called generalized random context picture grammars (grcpgs). In [8] we define grcpgs as grammars where the terminals are subsets of the Euclidean plane and the replacement of variables involves the building of functions that will eventually be applied to terminals. Context is again used to enable or inhibit the application of production rules.

In this form generalized random context picture grammars can be seen as a generalization of (context-free) collage grammars [9].

Iterated Function Systems (IFSs) are among the best-known methods for constructing fractals. In [8] we show that any picture sequence generated by an IFS can also be generated by a grcpg that uses forbidding context only. Moreover, since grcpgs use context to control the sequence in which functions are applied, they can generate a wider range of fractals or, more generally, pictures than IFSs [8].

Mutually Recursive Function Systems (MRFSs) are a generalization of IFSs. In [10] we show that any picture sequence generated by an IFS can also be generated by a grcpg that uses forbidding context only. Moreover, grcpgs can generate sequences of pictures that MRFSs cannot [10].

8 Future work

In this paper we give a summary of results for random context picture grammars and three of their more interesting subclasses, namely random permitting context picture grammars, random forbidding context picture grammars and table-driven context-free picture grammars.

It has been established that Tcfpgs are strictly weaker than rFcpgs. Moreover, it is known that Tcfpgs can generate a gallery that rPcpgs cannot, namely the gallery of approximations of the Sierpiński carpet. However, it is not known whether there exists a gallery that can be generated by an rPcpg, but not by any Tcfpg. Moreover, it is also not known if there is a gallery that can be generated by an rPcpg, but not by any rFcpg.

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