Orbit and bundle stratification of controllability and observability matrix pairs in StratiGraph

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Abstract. The canonical structures of controllability (A, B) and observability (A, C) matrix pairs associated with a state-space system are studied under small perturbations. We show how previous work for general matrix pencils can be applied to the stratification of controllability and observability pairs. We also present how the new results are used in StratiGraph, which is a software tool for computing and visualizing orbit and bundle closure hierarchies.

1 Introduction

Computing the canonical structure of a matrix pencil is a well known ill-posed problem. Small perturbations in the input data can dramatically change the canonical structure. For example, a square singular pencil becomes regular and multiple eigenvalues split apart. Nevertheless, degenerate canonical structures of matrix pencils appear in control applications, e.g., computing controllable subspaces and uncontrollable modes. Besides knowing the canonical structure of a system pencil associated with a state-space system, it is equally important to know its nearby canonical structures in order to explain the behaviour of the state-space system under small perturbations.

A stratification provides qualitative information about which structures are related to each other, which structures can be found near a specific matrix or matrix pencil, etc. The theory describing the complete stratification of orbits and bundles of general matrices and matrix pencils is presented by Edelman, Elmroth, and Kågström [2, 3]. Based on this theory, a software tool, StratiGraph, for computing and visualizing these hierarchies has been developed [11, 13, 4].

In line of this work, we now continue by considering the controllability and observability pairs (A, B) and (A, C) associated with the state-space system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1.1}$$

where $A \in \mathbb{C}^{n\times n},\ B \in \mathbb{C}^{n\times m},\ C \in \mathbb{C}^{p\times n}$ and $D \in \mathbb{C}^{p\times m}$. In this contribution, we show how the previous work for general matrix pencils can be applied to the stratification of controllability and observability pairs. We also present how the new results are used in StratiGraph.

2 Review of the general matrix pencil case

A general matrix pencil, $\mathbf{A} - \lambda\mathbf{B}$, where $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{m\times n}$ can have column and row minimal indices as well as finite and infinite eigenvalues (infinite if $\mathbf{B}$ is singular). Notice that all matrix pencils where $m \neq n$ are singular, which is the case in most control applications. Moreover, a general $m \times n$ matrix pencil can be transformed into *Kronecker Canonical Form*, KCF [7]:

$$P^{-1}(\mathbf{A} - \lambda\mathbf{B})Q = \mathrm{diag}(L_{\epsilon_1}, \ldots, L_{\epsilon_p}, J(\mu_1), \ldots, J(\mu_t), L^T_{\eta_1}, \ldots, L^T_{\eta_q}),$$

where P of size $m\times m$ and Q of size $n\times n$ are nonsingular. $J(\mu_1), \ldots, J(\mu_t)$ form the *regular structure* and are Jordan blocks of the finite and infinite eigenvalues:

$$J_j(\lambda_i) \equiv \begin{bmatrix} \mu_i - \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \mu_i - \lambda \end{bmatrix} \quad \text{and} \quad J_j(\infty) \equiv \begin{bmatrix} 1 & -\lambda & & \\ & \ddots & \ddots & \\ & & \ddots & -\lambda \\ & & & 1 \end{bmatrix}.$$

L_i and L_j^T correspond to the minimal indices of a singular pencil:

$$L_i \equiv \begin{bmatrix} -\lambda & 1 & & \\ & \ddots & \ddots & \\ & & -\lambda & 1 \end{bmatrix} \quad \text{and} \quad L_j^T \equiv \begin{bmatrix} -\lambda & & \\ 1 & \ddots & \\ & \ddots & -\lambda \\ & & 1 \end{bmatrix}.$$

An $i \times (i+1)$ block, L_i, is called a *right singular block* associated with a column minimal index i, and a $(j+1) \times j$ block, L_j^T, is called a *left singular block* associated with a row minimal index j. L_i has a right singular vector $x^T_{i+1} = [1\ \lambda\ \lambda^2\ \ldots\ \lambda^i]$ such that $L_i x_{i+1} = 0$ for any $\lambda \in \mathbb{C}$. Similarly, L_j^T has a left singular vector $y_{j+1} = [1\ \lambda\ \lambda^2\ \ldots \lambda^j]$ such that $y_{j+1}L_j^T = 0$ for any scalar λ. The right and left singular blocks form the *singular structure* of $\mathbf{A} - \lambda\mathbf{B}$.

2.1 Orbits and bundles

Two matrix pencils, $\mathbf{A}_1 - \lambda\mathbf{B}_1$ and $\mathbf{A}_2 - \lambda\mathbf{B}_2$, are said to be *strictly equivalent* if there exists non-singular matrices, P and Q, such that

$$\mathbf{A}_2 - \lambda\mathbf{B}_2 = P^{-1}(\mathbf{A}_1 - \lambda\mathbf{B}_1)Q.$$

The set of all equivalent pencils to $\mathbf{A} - \lambda\mathbf{B}$ defines the *equivalence orbit* of the pencil, i.e.,

$$\mathcal{O}(\mathbf{A} - \lambda\mathbf{B}) = \{P^{-1}(\mathbf{A} - \lambda\mathbf{B})Q|\det(P)\det(Q) \neq 0\}.$$

$\mathcal{O}(\mathbf{A} - \lambda\mathbf{B})$ consists of all pencils with the same eigenvalues and the same KCF as $\mathbf{A} - \lambda\mathbf{B}$. To be specific, these orbits of matrix pencils are manifolds in the $2mn$-dimensional space of $m \times n$ matrix pencils.

A *bundle* $\mathcal{B}(\mathbf{A} - \lambda\mathbf{B})$ is a union of orbits. If two pencils have the same Kronecker structure except that their distinct eigenvalues are different, they are said to be in the same bundle.

2.2 Codimension

The dimension of an orbit or bundle is equal to the dimension of its tangent space and is uniquely determined by the Kronecker structure. In practice, it is often more convenient to work with the dimension of the space complementary to the tangent space, denoted *codimension*.

The more degenerate the Kronecker structure of a pencil is, the smaller is the dimension and the larger is the codimension of its corresponding orbit and bundle. For the most generic pencil of size $m \times n$ ($m \neq n$), the orbit or the bundle spans the complete $2mn$-dimensional space, hence the codimension is zero. The most degenerate $m \times n$ ($m \neq n$) case is the zero pencil $0_{m\times n} - \lambda 0_{m\times n}$, which orbit and bundle both have codimension $2mn$.

The main difference between orbits and bundles is that the eigenvalues are not specified for a bundle, i.e., its tangent space spans one extra dimension for each distinct eigenvalue compared to the corresponding orbit. In conclusion, the codimension of a bundle is equal to the codimension of a corresponding orbit minus the number of distinct eigenvalues.

2.3 Integer partitions and stratification

Edelman, Elmroth, and Kågström [3] show how Kronecker structures can be represented as integer partitions such that the closure relations of the various orbits and bundles are revealed by applying a simple set of rules. The closure relations or the closure hierarchy form the *stratification* of Kronecker structures.

An integer partition $\kappa = (k_1,\ k_2,\ k_3,\ ...)$ such that $k_1 \geq k_2 \geq \ldots \geq 0$ is said to dominate another partition λ, i.e., $\kappa > \lambda$ if $k_1 + k_2 + \ldots + k_i \geq l_1 + l_2 + \ldots + l_i$ for $i = 1, 2, \ldots$, where $\lambda \neq \kappa$. Different partitions of an integer can in this way form a dominance ordering. If $\kappa > \lambda$, $\mathrm{sum}(\kappa) = \mathrm{sum}(\lambda)$ and there is no partition μ such that $\kappa > \mu > \lambda$, then κ is said to *cover* λ.

In the rules defined by Edelman, Elmroth, and Kågström, the integer partitions are illustrated as piles of coins in a table. An integer partition $\kappa = (k_1,\ k_2,\ \ldots,\ k_n)$ is represented as n piles of coins where pile i has k_i coins (see Figure 1a). The covering relation between two integer partitions can then easily by determined. If an integer partion μ can be obtained from κ by moving one coin in κ *one* column rightward or *one* row downward and μ remains monotonic decreasing (Figure 1b), then κ *covers* μ. This defines a *minimum rightward* coin move.

For a matrix pencil, the minimal column and row indices form the integer partitions $\mathcal{R} = (r_0, r_1, \ldots)$ and $\mathcal{L} = (l_0, l_1, \ldots)$, respectively. Here, r_i is the number of L_i blocks of size greater or equal to i. Similarly, l_j is the number of L_j^T blocks of size greater or equal to j. The sizes of the Jordan blocks in

(a) Coin table (b) Rightward move

Fig. 1. In (a) the integer partion (3, 2, 2, 1) is shown as a coin table and (b) shows a minimum rightward coin move, where (3, 2, 2, 1) becomes (2, 2, 2, 2) and hence (3, 2, 2, 1) covers (2, 2, 2, 2).

Weyr notation corresponding to each eigenvalue μ_i form the integer partitions $\mathcal{J}_{\mu_i} = (j_1^{(i)}, j_2^{(i)}, \ldots, j_{\max}^{(i)})$, i.e., $j_k^{(i)}$ is the number of Jordan blocks of size greater or equal to k. The rules to apply to get the stratification of a matrix pencil is shown in Theorem 1.

Theorem 1. [3] *Given the structure integer partitions $\mathcal{L}, \mathcal{R}$ and $\mathcal{J}_{\mu_i}$ of $\mathbf{A} - \lambda\mathbf{B}$, the following if-and-only-if rules find $\widetilde{\mathbf{A}} - \lambda\widetilde{\mathbf{B}}$ fulfilling orbit or bundle covering relations with $\mathbf{A} - \lambda\mathbf{B}$.*

$\mathcal{O}(\mathbf{A} - \lambda\mathbf{B})$ covers $\mathcal{O}(\widetilde{\mathbf{A}} - \lambda\widetilde{\mathbf{B}})$:

(1) *Minimum* rightward *coin move in $\mathcal{R}$ (or $\mathcal{L}$).*
(2) *If the rightmost column in $\mathcal{R}$ (or $\mathcal{L}$) is one single coin, move that coin to a new rightmost column of some $\mathcal{J}_{\mu_i}$ (which may be empty initially).*
(3) *Minimum* leftward *coin move in any $\mathcal{J}_{\mu_i}$.*
(4) *Let k denote the total number of coins in all of the longest (= lowest) rows from all of the $\mathcal{J}_{\mu_i}$. Remove these k coins, add one more coin to the set, and distribute $k+1$ coins to r_p, $p = 0, \ldots, t$ and l_q, $q = 0, \ldots, k-t-1$ such that at least all non-zero columns of $\mathcal{R}$ and $\mathcal{L}$ are given coins.*

Rules 1 and 2 may not make coin moves that affect r_0 (or l_0).

$\mathcal{B}(\mathbf{A} - \lambda\mathbf{B})$ covers $\mathcal{B}(\widetilde{\mathbf{A}} - \lambda\widetilde{\mathbf{B}})$:

(1) *Same as rule 1 to the left.*
(2) *Same as rule 2 to the left, except it is allowed only to start a new set corresponding to a new eigenvalue (i.e., no appending to nonempty sets).*
(3) *Same as rule 3 to the left.*
(4) *Same as rule 4 to the left, but apply only if there is just one eigenvalue in the KCF or if all eigenvalues have at least two Jordan blocks.*
(5) *Let any pair of eigenvalues coalesce, i.e., take the union of their sets of coins.*

We remark that several orbits (or bundles) in a closure hierarchy can have the same codimension, which corresponds to branches in the hierarchy. However, an orbit (or bundle) structure can never be covered by a less or equally generic structure. This implies that structures within a branch of a closure hierarchy can be ordered by their codimensions (or dimensions).

2.4 Closure hierarchies as a graph representation in StratiGraph

The closure hierarchy of canonical structures of an orbit (or bundle) can be represented as a connected graph, where the nodes in the graph correspond to different canonical structures in the hierarchy, and the edges represent the covering relations. This representation is used in StratiGraph. Several structures in different branches of the closure hierarchy can have the same codimension and are then aligned on the same horizontal level. A screen-shot of a StratiGraph graph is shown in Figure 2.

3 Stratification of matrix pairs

A state-space system (1.1) can be represented and analyzed in terms of a *system pencil*

$$\mathbf{S}(\lambda) = \mathbf{A} - \lambda \mathbf{B}, \quad \text{where} \quad \mathbf{A} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \; \mathbf{B} = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}.$$

Consequently, the system pencils associated with the controllability pair (A, B) and the observability pair (A, C) are

$$\mathbf{S}_{\mathrm{C}}(\lambda) = \begin{bmatrix} A & B \end{bmatrix} - \lambda \begin{bmatrix} I_n & 0 \end{bmatrix}, \tag{3.2}$$

and

$$\mathbf{S}_{\mathrm{O}}(\lambda) = \begin{bmatrix} A \\ C \end{bmatrix} - \lambda \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \tag{3.3}$$

respectively. Notice that the system pencils $\mathbf{S}_{\mathrm{C}}(\lambda)$ and $\mathbf{S}_{\mathrm{O}}(\lambda)$ are special cases of $\mathbf{S}(\lambda)$. Due to the special structure of the λ-part matrix of $\mathbf{S}_{\mathrm{C}}(\lambda)$, the controllability system pencil can only have right singular blocks L_i and finite eigenvalues in its KCF. Similarly, the λ-part matrix of $\mathbf{S}_{\mathrm{O}}(\lambda)$ has full column rank and it can only have left singular blocks L_j^T and finite eigenvalues in its KCF.

In the following we consider the orbit and bundle for *Γ-equivalence* of matrix pairs [14]. Γ-equivalence for a controllability pair (A, B) is defined as

$$P \begin{bmatrix} A - \lambda I & B \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ R & Q^{-1} \end{bmatrix} = \begin{bmatrix} P(A - \lambda I)P^{-1} + PBR & PBQ^{-1} \end{bmatrix}, \tag{3.4}$$

and for the observability pair (A, C) as

$$\begin{bmatrix} P & S \\ 0 & T \end{bmatrix} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} P^{-1} = \begin{bmatrix} P(A - \lambda I)P^{-1} + SCP^{-1} \\ TCP^{-1} \end{bmatrix}, \tag{3.5}$$

where $P \in \mathbb{C}^{n \times n}$, $Q \in \mathbb{C}^{m \times m}$, $T \in \mathbb{C}^{p \times p}$,

$$\begin{bmatrix} P^{-1} & 0 \\ R & Q^{-1} \end{bmatrix} \text{ and } \begin{bmatrix} P & S \\ 0 & T \end{bmatrix}$$

are nonsingular, and $R \in \mathbb{C}^{m \times n}$ and $S \in \mathbb{C}^{n \times p}$.

Both necessary and sufficient conditions for closures of matrix pairs have been studied in [8–10]. In [9], also the necessary conditions for cover relations of matrix pencils with no minimal row indices has been derived. From [9, 10] and [3] it is possible to derive sufficient as well as necessary conditions for covering relations of matrix pairs.

Expressed in coin moves, a less generic matrix pair can be obtained by the rules of the following theorem.

Theorem 2. [3, 12] *Given the structure integer partitions $\mathcal{L}, \mathcal{R}$ and $\mathcal{J}_{\mu_i}$ of (A, B) or (A, C), the following if-and-only-if rules find $(\widetilde{A}, \widetilde{B})$ or $(\widetilde{A}, \widetilde{C})$ fulfilling orbit or bundle covering relations with (A, B) or (A, C), respectively.*

$\mathcal{O}(A, B)$ covers $\mathcal{O}(\widetilde{A}, \widetilde{B})$ (or $\mathcal{O}(A, C)$ covers $\mathcal{O}(\widetilde{A}, \widetilde{C})$):

1. *Minimal* rightward *coin move in $\mathcal{R}$ (or $\mathcal{L}$).*
2. *If the rightmost column in $\mathcal{R}$ (or $\mathcal{L}$) is one single coin, move that coin as a new rightmost column of some $\mathcal{J}_{\mu_i}$ (which may be empty initially).*
3. *Minimal* leftward *coin move in any $\mathcal{J}_{\mu_i}$.*

Rules 1 and 2 may not make coin moves that affect r_0 (or l_0).

$\mathcal{B}(A, B)$ covers $\mathcal{B}(\widetilde{A}, \widetilde{B})$ (or $\mathcal{B}(A, C)$ covers $\mathcal{B}(\widetilde{A}, \widetilde{C})$):

1. *Same as rule 1 to the left.*
2. *Same as rule 2 to the left, except it is allowed only to start a new set corresponding to a new eigenvalue (i.e., no appending to nonempty sets).*
3. *Same as rule 3 to the left.*
4. *Let any pair of eigenvalues coalesce, i.e., take the union of their sets of coins.*

The major difference between the rules for matrix pencils and matrix pairs, is that rule 4 (both for orbits and bundles) in Theorem 1 does not apply to matrix pairs. The rule does not exist since there is only one type of singular blocks (L_i or L_j^T) in each matrix pair type. Moreover, in rules 1 and 2 of Theorem 2, the (A, B) pair applies to the $\mathcal{R}$ partition only and the (A, C) pair applies to the $\mathcal{L}$ partition only.

Corollary 1. [12] *$\mathcal{O}(A, B)$ covers $\mathcal{O}(\widetilde{A}, \widetilde{B})$ (or $\mathcal{O}(A, C)$ covers $\mathcal{O}(\widetilde{A}, \widetilde{C})$) if and only if $(\widetilde{A}, \widetilde{B})$ (or $(\widetilde{A}, \widetilde{C})$) can be obtained from (A, B) (or (A, C)) by using one of the rules in the left part of Theorem 2.*

Corollary 2. [12] *$\mathcal{B}(A, B)$ covers $\mathcal{B}(\widetilde{A}, \widetilde{B})$ (or $\mathcal{B}(A, C)$ covers $\mathcal{B}(\widetilde{A}, \widetilde{C})$) if and only if $(\widetilde{A}, \widetilde{B})$ (or $(\widetilde{A}, \widetilde{C})$) can be obtained from (A, B) (or (A, C)) by using one of the rules in the right part of Theorem 2.*

The codimension of the orbit for matrix pairs can be calculated as the sum of separate codimensions [1, 6]:

$$\operatorname{cod}(A, B) = c_{Jor} + c_{Right} + c_{Jor,Sing}, \tag{3.6}$$

and

$$\operatorname{cod}(A, C) = c_{Jor} + c_{Left} + c_{Jor,Sing}. \tag{3.7}$$

The sums come from the interaction between the Jordan blocks, the right/left singular blocks ($L_j \leftrightarrow L_k$ or $L_j^T \leftrightarrow L_k^T$), and from the interaction of the Jordan structure with the singular blocks. Let $s_1(\mu_i) \geq s_2(\mu_i) \geq \cdots \geq s_{g_i}(\mu_i)$ denote the sizes of the Jordan blocks corresponding to eigenvalue μ_i with g_i blocks, $i = 1, \ldots, t$. Then the separate codimensions are given as

$$c_{Jor} = \sum_{i=1}^{t} \sum_{j=1}^{g_i} (2j-1) s_j(\mu_i) = \sum_{i=1}^{t} (s_1(\mu_i) + 3s_2(\mu_i) + \cdots + (2g_i - 1) s_{g_i}(\mu_i)),$$

$$c_{Right} = \sum_{j>k}(j-k-1), \quad c_{Left} = \sum_{j>k}(j-k-1), \text{ and}$$

$$c_{Jor,Sing} = (\text{size of complete regular part}) \cdot (\text{number of singular blocks}).$$

The codimension of an associated bundle is equal to the codimension of the orbit minus the number of distinct eigenvalues.

The generic Kronecker structure of the controllability pair (A, B) has $\mathcal{R} = (r_0, \ldots, r_\alpha, r_{\alpha+1})$ where $r_0 = \cdots = r_\alpha = m$, $r_{\alpha+1} = n \mod m$, and $\alpha = \lfloor n/m \rfloor$. For the observability pair (A, C) the generic case has $\mathcal{L} = (l_0, \ldots, l_\alpha, l_{\alpha+1})$ where $l_0 = \cdots = l_\alpha = p$, $l_{\alpha+1} = n \mod p$, and $\alpha = \lfloor n/p \rfloor$. The most degenerated case of $\mathbf{S}_\mathrm{C}(\lambda)$ has m L_0 blocks and n Jordan blocks of size 1×1 corresponding to an eigenvalue of multiplicity n. Similarly, $\mathbf{S}_\mathrm{O}(\lambda)$ has m L_0^T blocks and n 1×1 Jordan blocks. In other words, the most generic cases correspond to completely controllable and observable systems, while the most degerate cases correspond to systems with n uncontrollable and n unobservable multiple modes, respectively.

3.1 A 4×2 observability matrix pair

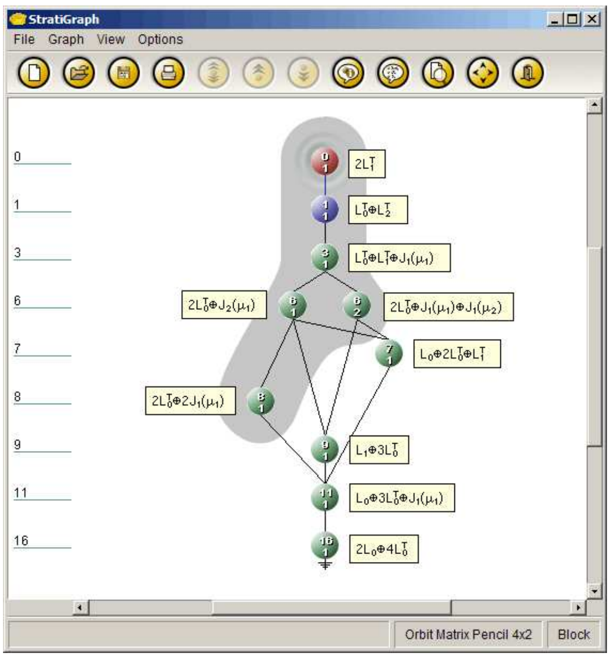

Fig. 2. Screen-shot from StratiGraph visualizing the complete stratification of the orbit to a general 4×2 matrix pencil. The grayed area marks the structures with no right singular blocks.

For illustration we consider the stratification of the orbit of a small 4×4 system pencil with two states, two inputs and two outputs:

$$\mathbf{S}(\lambda) = \mathbf{A} - \lambda \mathbf{B} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} - \lambda \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix},$$

where $A, B, C \in \mathbb{C}^{2\times 2}$.

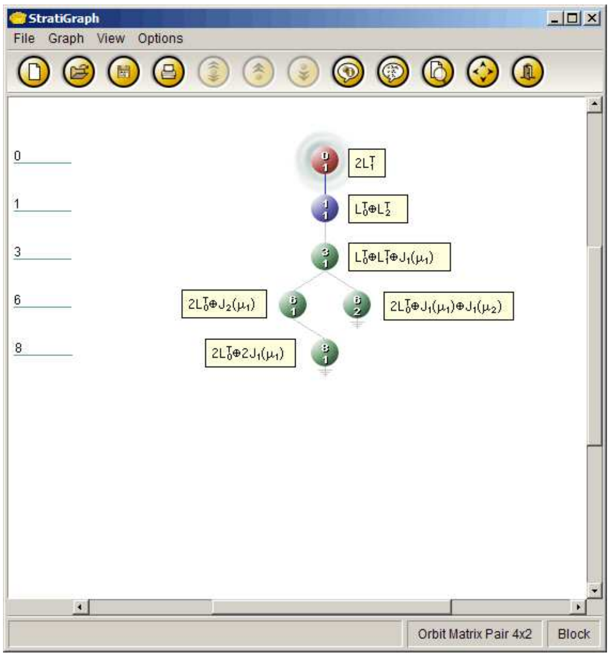

Fig. 3. Screen-shot from StratiGraph visualizing the complete stratification of the orbit corresponding to a 4×2 observability matrix pair (A, C) with two states and two outputs.

The orbit stratification of a 4×4 general matrix pencil is a graph with 47 different structures and does not consider the special structure of the controllability and observability pairs. We start by considering the observability pair (A, C). The observability system pencil

$$\mathbf{S}_{\mathrm{O}}(\lambda) = \begin{bmatrix} A \\ C \end{bmatrix} - \lambda \begin{bmatrix} I_2 \\ 0 \end{bmatrix},$$

is now 4×2. The stratification of the orbit of a general 4×2 matrix pencil has only 10 structures, illustrated in Figure 2, which shows the closure hierarchy graph computed and visualized by StratiGraph. Still we have not used the special structure of $\mathbf{S}_{\mathrm{O}}(\lambda)$. Considering that $\mathbf{S}_{\mathrm{O}}(\lambda)$ has full column rank, the pencil can have no right singular blocks. From Figure 2, we see that in the more degenerate structures, not only left singular blocks appear but also right singular blocks that we know can not exist.

StratiGraph has recently been extended with built-in support for matrix pairs (A, B) and (A, C). In Figure 3, the stratification of the same problem size as in Figure 2 is shown, but now as a matrix pair (A, C) when the rules in Theorem 2 is used. The closure hierarchy graph of $\mathbf{S}_{\mathrm{O}}(\lambda)$ is identical to the grayed part of the graph shown in Figure 2, i.e., the part of the graph with no right singular blocks.

The result is very similar for the controllability matrix pair (A, B), but compared to a general 2×4 matrix pencil, the resulting graph has no structures with left singular blocks. This is the case both when looking at orbits as well as bundles.

In conclusion, the incorporation of the stratification of observability and controllability pairs into StratiGraph makes it much easier to view and understand the qualitative behavior of such pairs under small perturbations. Ongoing work include the study of matrix triplets and quadruples and the incorporation of quantitative information in StratiGraph, providing computable bounds on the distance to nearby structures in a closure hierarchy [5].

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