

# STRATIFICATION OF CONTROLLABILITY AND OBSERVABILITY PAIRS — THEORY AND USE IN APPLICATIONS\*

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**Abstract.** Cover relations for orbits and bundles of controllability and observability pairs associated with linear time-invariant systems are derived. The cover relations are combinatorial rules acting on integer sequences, each representing a subset of the Jordan and singular Kronecker structures of the corresponding system pencil. By representing these integer sequences as coin piles, the derived stratification rules are expressed as minimal coin moves between and within these piles, which satisfy and preserve certain monotonicity properties. The stratification theory is illustrated with two examples from systems and control applications, a mechanical system consisting of a thin uniform platform supported at both ends by springs, and a linearized Boeing 747 model. For both examples, nearby uncontrollable systems are identified as subsets of the complete closure hierarchy for the associated system pencils.

**Key words.** Stratification, matrix pairs, controllability, observability, robustness, Kronecker structures, orbit, bundle, closure hierarchy, cover relations, StratiGraph.

**1. Introduction.** Computing the canonical structure of a linear time-invariant (LTI) system,  $\dot{x}(t) = Ax(t) + Bu(t)$  with states  $x(t)$  and inputs  $u(t)$ , is an ill-posed problem, i.e., small changes in the input data matrices  $A$  and  $B$  may drastically change the computed canonical structure of the associated system pencil  $[A - \lambda I \ B]$  (e.g., see [13]). Besides knowing the canonical structure, it is equally important to be able to identify nearby canonical structures in order to explain the behavior and possibly determining the robustness of a state-space system under small perturbations. For example, a state-space system which is found to be controllable may be very close to an uncontrollable one, and can therefore by only a small change in some data, e.g., due to round-off or measurement errors, become uncontrollable. If the LTI system considered and all nearby systems in a given neighborhood are controllable, the system is called *robustly controllable* (e.g., see [46]).

The qualitative information about nearby linear systems is revealed by the theory of stratification for the corresponding system pencil. A *stratification* shows which canonical structures are near to each other (in the sense of small perturbations) and their relation to other structures, i.e., the theory reveals the closure hierarchy of orbits and bundles of canonical structures. A cover relation guarantees that two canonical structures are nearest neighbours in the closure hierarchy.

For square matrices, Arnold [1] examined nearby structures by small perturbations using versal deformations. For matrix pencils, Elmroth and Kågström [23] first investigated the set of 2-by-3 matrix pencils and later extended the theory, in collaboration with Edelman, to general matrices and matrix pencils [17, 18]. In line of this work, the theory has further been developed in [21], and for matrix pairs in [20, 42]. Several other people have worked on the theory of stratifications and similar topics, and we refer to [2, 27, 31, 35, 49] and references there in. Furthermore, the related topic distance to uncontrollability has recently been studied in, e.g., [6, 22, 30, 33, 34, 46].

In this paper, we derive the *cover relations* for independent controllability and

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observability pairs associated with LTI systems. These relations are combinatorial rules acting on integer sequences, each representing a subset of the Jordan and singular Kronecker structures (canonical form) of the corresponding system pencil. By following [17, 18], and representing these integer sequences as coin piles, the derived stratification rules are expressed as simple coin moves between and within these piles. Besides, only coin moves that satisfy and preserve certain monotonicity properties of the integer sequences are valid moves.

Before we go into further details, we outline the contents of the rest of the paper. In Section 2, some linear systems background, including matrix pencil representations, are presented. In addition, a subsection introduces minimum coin moves for piles of coins representing integer partitions that frequently appear in the covering rules. Section 3 gives a concise presentation of the Kronecker canonical form (KCF) of a general matrix pencil and its invariants, as well as the Brunovsky canonical form for various system pencils. In Section 4, system pencils for matrix pairs are considered. Concepts introduced include orbits and bundles for controllability and observability pairs, matrix representations for associated tangent spaces, and their codimensions expressed in terms of the KCF invariants. Equipped with all these concepts and notation, Section 5 is devoted to the stratification theory, focusing on the derivation of cover relations for matrix pair orbits and bundles. In Section 6, we illustrate the stratification theory by considering two examples from systems and control applications, a mechanical system consisting of a thin uniform platform supported at both ends by springs [44], and a linearized Boeing 747 model [51]. For both examples, we identify nearby uncontrollable systems as subsets of the complete closure hierarchy for the associated system pencils.

Following [23, 18], we present stratifications as graphs where each node represents an orbit or a bundle of a canonical structure and an edge represents a covering relation. A graph is organized with the most generic structure(s) at the top and other structures further down, ordered by increasing degeneracy (increasing codimension). Figure 1.1 illustrates how to interpret such a graph, assuming that each node represents the orbit of some canonical structure.

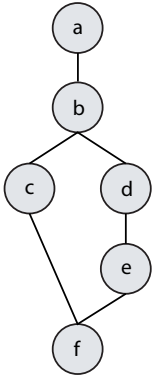


FIG. 1.1. A graph presenting a hypothetical closure hierarchy, where letters ( $a - f$ ) represent some canonical structures, the nodes represent orbits of these structures, and the edges represent covering relations.

The topmost node shows the structure denoted  $a$  as the most generic structure. The edge to the node  $b$  illustrates that  $a$  covers  $b$ , i.e., the orbit of  $b$  is in the closure

of that of  $a$  and there are no other structures between them in the closure hierarchy. Notably, all structures in the closure of  $b$  are also in the closure of  $a$ , although there are no covering relations between  $a$  and these structures since  $b$  appears between them in the hierarchy. Continuing downwards,  $b$  covers both  $c$  and  $d$  and there is no covering relation between  $c$  and  $d$ . Further down, the orbit of  $e$  is in the closure of that of  $d$  but not in the closure of  $c$ 's orbit. The most degenerate structure is  $f$ , which is covered by both  $c$  and  $e$ , actually showing that  $f$ 's orbit is in the intersection of the orbits of  $c$  and  $e$ . In this example,  $f$  is the most degenerate structure, whose orbit is in the closure of all other orbits.

In Section 6, we make use of this type of graphs to illustrate closure hierarchies. The graphs presented are generated with StratiGraph [21, 38, 40, 41], which is a software tool for determining and presenting closure hierarchies based on the theory in [17, 18, 42]. The current version of StratiGraph (v. 2.1) has support for stratification of matrices, matrix pencils, and controllability and observability pairs. The theory of the latter is presented and illustrated in this paper.

**2. Background and notation.** A *linear time-invariant, finite dimensional system* (LTI system) is in continuous time represented as a state-space model by a system of the differential equations

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t),\end{aligned}\tag{2.1}$$

where  $A \in \mathbb{C}^{n \times n}$ ,  $B \in \mathbb{C}^{n \times m}$ ,  $C \in \mathbb{C}^{p \times n}$  and  $D \in \mathbb{C}^{p \times m}$ . Such a state-space system is in short form represented by the *quadruple of matrices*  $(A, B, C, D)$ .

System (2.1) is said to be *controllable* if there exists an input signal  $u(t)$ ,  $t_0 \leq t \leq t_f$ , that takes every state variable from an initial state  $x(t_0)$  to a desired final state  $x(t_f)$  in finite time. Otherwise it is said to be *uncontrollable*. The dual concept of controllability is observability. System (2.1) is said to be *observable* if it is possible to find the initial state  $x(t_0)$  from the input signal  $u(t)$  and the output signal  $y(t)$  measured over a finite interval  $t_0 \leq t \leq t_f$ . Otherwise it is said to be *unobservable*.

The controllability and observability of a system only depend on the matrix pairs  $(A, B)$  and  $(A, C)$ , respectively, associated with the particular systems

$$\dot{x}(t) = Ax(t) + Bu(t), \quad \text{and} \quad \begin{aligned}\dot{x}(t) &= Ax(t), \\ y(t) &= Cx(t),\end{aligned}$$

of (2.1). The matrix pairs  $(A, B)$  and  $(A, C)$  are referred to as the *controllability* and *observability pairs*, respectively.

**2.1. The pencil representation.** The set of matrices of the form  $G - \lambda H$  with  $\lambda \in \mathbb{C}$  corresponds to a general *matrix pencil*, where the two complex matrices  $G$  and  $H$  are of size  $m_p \times n_p$ . Notice that all matrix pencils where  $m_p \neq n_p$  are singular, which is the case in most control applications.

A state-space system (2.1) can also be represented and analyzed in terms of a matrix pencil, which in this special form is called a *system pencil*,  $\mathbf{S}(\lambda)$ . In contrary to a general matrix pencil, a system pencil emphasizes the structure of the system. The associated system pencil for the state-space system (2.1) is

$$\mathbf{S}(\lambda) = G - \lambda H = \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \lambda \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix},\tag{2.2}$$

where  $G$  and  $H$  are of size  $(n+p) \times (n+m)$  and consequently  $m_p = n+p$  and  $n_p = n+m$ . The corresponding system pencils for the controllability and observability pairs are

$$\mathbf{S}_C(\lambda) = [A \ B] - \lambda [I_n \ 0], \quad \text{and} \quad \mathbf{S}_O(\lambda) = \begin{bmatrix} A \\ C \end{bmatrix} - \lambda \begin{bmatrix} I_n \\ 0 \end{bmatrix}.$$

In the rest of the paper, we are mainly only considering the controllability and observability pairs and their associated system pencils.

**2.2. Integer partitions and coins.** We give a brief introduction to integer partitions and minimum coin moves, which are used to represent the invariants of the matrix and system pencils and to define the stratification rules.

An *integer partition*  $\kappa = (\kappa_1, \kappa_2, \dots)$  of an integer  $K$  is a monotonically decreasing sequence of integers ( $\kappa_1 \geq \kappa_2 \geq \dots \geq 0$ ) where  $\kappa_1 + \kappa_2 + \dots = K$ . We denote the sum  $\kappa_1 + \kappa_2 + \dots$  as  $\sum \kappa$ . The *union*  $\tau = (\tau_1, \tau_2, \dots)$  of two integer partitions  $\kappa$  and  $\nu$  is defined as  $\tau = \kappa \cup \nu$  where  $\tau_1 \geq \tau_2 \geq \dots$ . The *difference*  $\tau$  of two integer partitions  $\kappa$  and  $\nu$  is defined as  $\tau = \kappa \setminus \nu$ , where  $\tau$  includes the elements from  $\kappa$  except elements existing in both  $\kappa$  and  $\nu$ , which are removed. Furthermore, the *conjugate partition* of  $\kappa$  is defined as  $\nu = \text{conj}(\kappa)$ , where  $\nu_i$  is equal to the number of integers in  $\kappa$  that is equal or greater than  $i$ , for  $i = 1, 2, \dots$ .

If  $\nu$  is an integer partition, not necessarily of the same integer  $K$  as  $\kappa$ , and  $\kappa_1 + \dots + \kappa_i \geq \nu_1 + \dots + \nu_i$  for  $i = 1, 2, \dots$ , then  $\kappa \geq \nu$ . When  $\kappa \geq \nu$  and  $\kappa \neq \nu$  then  $\kappa > \nu$ . If  $\kappa, \nu$  and  $\tau$  are integer partitions of the same integer  $K$  and there does not exist any  $\tau$  such that  $\kappa > \tau > \nu$  where  $\kappa > \nu$ , then  $\kappa$  *covers*  $\nu$ . It follows that  $\kappa$  covers  $\nu$  if and only if  $\kappa > \nu$  and  $\text{conj}(\kappa) < \text{conj}(\nu)$ . A weaker definition of cover is *adjacent* [11, 35], where  $\kappa$  and  $\nu$  can be partitions of different integers. We say that  $\kappa > \nu$  are adjacent partitions if either  $\kappa$  covers  $\nu$  or if  $\kappa = \nu \cup (1)$ .

An integer partition  $\kappa = (\kappa_1, \dots, \kappa_n)$  can also be represented by  $n$  piles of coins, where the first pile has  $\kappa_1$  coins, the second  $\kappa_2$  coins and so on. An integer partition  $\kappa$  *covers*  $\nu$  if  $\nu$  can be obtained from  $\kappa$  by moving one coin *one* column rightward or *one* row downward, and keep  $\kappa$  monotonically decreasing. Or equivalently, an integer partition  $\kappa$  *is covered by*  $\tau$  if  $\tau$  can be obtained from  $\kappa$  by moving one coin *one* column leftward or *one* row upward, and keep  $\kappa$  monotonically decreasing. These two types of coin moves are defined in [18] and called *minimum rightward* and *minimum leftward coin moves*, respectively (see Figure 2.1).

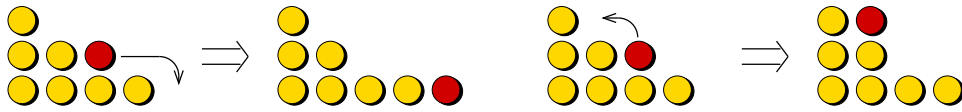


FIG. 2.1. *Minimum rightward and leftward coin moves illustrate that  $\kappa = (3, 2, 2, 1)$  covers  $\nu = (3, 2, 1, 1, 1)$  and  $\kappa = (3, 2, 2, 1)$  is covered by  $\tau = (3, 3, 1, 1)$ .*

**3. Canonical forms and invariants.** In the following, we introduce the Kronecker canonical form (KCF) of a general matrix pencil and its invariants in terms of integer sequences, as well as the Brunovsky canonical form for various system pencils.

**3.1. Kronecker canonical form.** Any general  $m_p \times n_p$  matrix pencil  $G - \lambda H$  can be transformed into *Kronecker canonical form* (KCF) in terms of an equivalence

transformation with two nonsingular matrices  $U$  and  $V$  [26]:

$$\begin{aligned} U(G - \lambda H)V^{-1} \\ = \text{diag}(L_{\epsilon_1}, \dots, L_{\epsilon_{r_0}}, J(\mu_1), \dots, J(\mu_q), N_{s_1}, \dots, N_{s_{g_\infty}}, L_{\eta_1}^T, \dots, L_{\eta_{l_0}}^T), \end{aligned} \quad (3.1)$$

where  $J(\mu_i) = \text{diag}(J_{h_1}(\mu_i), \dots, J_{h_{g_i}}(\mu_i))$ ,  $i = 1, \dots, q$ . The blocks  $J_{h_k}(\mu_i)$  are  $h_k \times h_k$  *Jordan blocks* associated with each distinct finite eigenvalue  $\mu_i$  and the blocks  $N_{s_k}$  are  $s_k \times s_k$  *Jordan blocks* for matrix pencils associated with the infinite eigenvalue. These two types of blocks constitute the *regular part* of a matrix pencil and are defined by

$$J_{h_k}(\mu_i) = \begin{bmatrix} \mu_i - \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \mu_i - \lambda & 1 \\ & & & \mu_i - \lambda \end{bmatrix}, \quad \text{and} \quad N_{s_k} = \begin{bmatrix} 1 & -\lambda & & \\ & \ddots & \ddots & \\ & & 1 & -\lambda \\ & & & 1 \end{bmatrix}.$$

If  $m_p \neq n_p$  or  $\det(G - \lambda H) \equiv 0$  for all  $\lambda \in \mathbb{C}$ , then  $r_0 \geq 1$  and/or  $l_0 \geq 1$  and the matrix pencil also includes a *singular part* which consists of the  $r_0$  *right singular blocks*  $L_{\epsilon_k}$  of size  $\epsilon_k \times (\epsilon_k + 1)$  and the  $l_0$  *left singular blocks*  $L_{\eta_k}^T$  of size  $(\eta_k + 1) \times \eta_k$ :

$$L_{\epsilon_k} = \begin{bmatrix} -\lambda & 1 & & \\ & \ddots & \ddots & \\ & & -\lambda & 1 \end{bmatrix}, \quad \text{and} \quad L_{\eta_k}^T = \begin{bmatrix} -\lambda & & & \\ 1 & \ddots & & \\ & \ddots & -\lambda & \\ & & & 1 \end{bmatrix}.$$

$L_0$  and  $L_0^T$  blocks are of size  $0 \times 1$  and  $1 \times 0$ , respectively, and each of them contributes with a column or row of zeros.

In general, a block diagonal matrix  $A = \text{diag}(A_1, A_2, \dots, A_q)$  with  $q$  blocks can also be represented as a direct sum

$$A \equiv A_1 \oplus A_2 \oplus \dots \oplus A_q \equiv \bigoplus_{k=1}^q A_k.$$

Using this notation, the KCF (3.1) can compactly be rewritten as

$$U(G - \lambda H)V^{-1} \equiv \mathbb{L} \oplus \mathbb{L}^T \oplus \mathbb{J}(\mu_1) \oplus \dots \oplus \mathbb{J}(\mu_q) \oplus \mathbb{N},$$

where

$$\mathbb{L} = \bigoplus_{k=1}^{r_0} L_{\epsilon_k}, \quad \mathbb{L}^T = \bigoplus_{k=1}^{l_0} L_{\eta_k}^T, \quad \mathbb{J}(\mu_i) = \bigoplus_{k=1}^{g_i} J_{h_k}(\mu_i), \quad \text{and} \quad \mathbb{N} = \bigoplus_{k=1}^{g_\infty} N_{s_k}.$$

Without loss of generality, we order the blocks of the KCF in the direct sum notation so that the singular blocks ( $\mathbb{L}$  and  $\mathbb{L}^T$ ) appear first.

**3.2. Invariants of matrix pencils.** The matrix pencil characteristics can equivalently be expressed in terms of column/row minimal indices and finite/infinite elementary divisors. Two matrix pencils are strictly equivalent if and only if they have the same minimal indices and elementary divisors or, equivalently, if they have the same KCF, i.e., the same  $L$ ,  $L^T$ ,  $J$  and  $N$  blocks.

The four invariants are defined as follows [26]:

(i) The *column (right) minimal indices* are  $\epsilon = (\epsilon_1, \dots, \epsilon_{r_0})$ , where  $\epsilon_1 \geq \epsilon_2 \geq \dots \geq \epsilon_{r_1} > \epsilon_{r_1+1} = \dots = \epsilon_{r_0} = 0$  define the sizes of the  $L_{\epsilon_k}$  blocks,  $\epsilon_k \times (\epsilon_k + 1)$ .

From the conjugate partition  $(r_1, \dots, r_{\epsilon_1}, 0, \dots)$  of  $\epsilon$  we define the integer partition  $\mathcal{R}(G - \lambda H) = (r_0) \cup (r_1, \dots, r_{\epsilon_1})$ .

(ii) The *row (left) minimal indices* are  $\eta = (\eta_1, \dots, \eta_l)$ , where  $\eta_1 \geq \eta_2 \geq \dots \geq \eta_l > \eta_{l+1} = \dots = \eta_0 = 0$  define the sizes of the  $L_{\eta_k}^T$  blocks,  $(\eta_k + 1) \times \eta_k$ . From the conjugate partition  $(l_1, \dots, l_{\eta_1}, 0, \dots)$  of  $\eta$  we define the integer partition  $\mathcal{L}(G - \lambda H) = (l_0) \cup (l_1, \dots, l_{\eta_1})$ .

(iii) The *finite elementary divisors* are of the form  $(\lambda - \mu_i)^{h_1^{(i)}}, \dots, (\lambda - \mu_i)^{h_{g_i}^{(i)}}$ , with  $h_1^{(i)} \geq \dots \geq h_{g_i}^{(i)} \geq 1$  for each of the  $q$  distinct finite eigenvalue  $\mu_i$ ,  $i = 1, \dots, q$ . Here,  $g_i$  is the geometric multiplicity of  $\mu_i$  and the sum of all  $h_k^{(i)}$  for  $k = 1, \dots, g_i$  is the algebraic multiplicity of  $\mu_i$ . For each distinct eigenvalue  $\mu_i$  we introduce the integer partition  $h_{\mu_i} = (h_1^{(i)}, \dots, h_{g_i}^{(i)})$  which is known as the *Segre characteristics*. These characteristics correspond to the sizes  $h_k^{(i)} \times h_k^{(i)}$  of the  $J_{h_k}(\mu_i)$  blocks (the largest first). The conjugate partition  $\mathcal{J}_{\mu_i}(G - \lambda H) = (j_1, j_2, \dots)$  of  $h_{\mu_i}$ , is the *Weyr characteristics* of  $\mu_i$ .

(iv) The *infinite elementary divisors* are of the form  $\rho^{s_1}, \rho^{s_2}, \dots, \rho^{s_{g_\infty}}$ , with  $s_1 \geq \dots \geq s_{g_\infty} \geq 1$ , where  $g_\infty$  is the geometric multiplicity of the infinite eigenvalue and the sum of all  $s_k$  for  $k = 1, \dots, g_\infty$  is the algebraic multiplicity. Similarly to case (iii), the integer partition  $s = (s_1, \dots, s_{g_\infty})$  is the *Segre characteristics* for the infinite eigenvalue, which correspond to the sizes  $s_k \times s_k$  of the  $N_{s_k}$  blocks. The conjugate partition  $\mathcal{N}(G - \lambda H) = (n_1, n_2, \dots)$  of  $s$ , is the *Weyr characteristics* of the infinite eigenvalue.

When it is clear from context, we use the abbreviated notation  $\mathcal{R}$ ,  $\mathcal{L}$ ,  $\mathcal{J}$ , and  $\mathcal{N}$ , for the above defined integer partitions corresponding to the right and left singular structures, and the Jordan structures of the finite and infinite eigenvalues, respectively. In the following, these integer partitions are referred to as *structure integer partitions*.

The system pencils  $\mathbf{S}(\lambda)$ ,  $\mathbf{S}_C(\lambda)$ , and  $\mathbf{S}_O(\lambda)$ , can also be expressed in terms of the above invariants and their associated structure integer partitions. However, in general their corresponding invariants are different. For example, the system pencil  $\mathbf{S}_C(\lambda)$  of a completely controllable system associated with the pair  $(A, B)$  can only have  $L$  blocks in its KCF while  $\mathbf{S}(\lambda)$  (2.2) may have both types of singular invariants (blocks) as well as eigenvalues in its KCF.

**3.3. Brunovsky canonical form.** When considering canonical forms of the system pencils  $\mathbf{S}_C(\lambda)$  and  $\mathbf{S}_O(\lambda)$  associated with pairs of matrices, we are (mainly) interested in canonical forms obtained from structure-preserving equivalence transformations. One such example is the Brunovsky canonical form. This canonical form explicitly reveals the system characteristics from the system pencils. This is in contrast to the KCF, which destroys the special block structure of  $\mathbf{S}_C(\lambda)$  and  $\mathbf{S}_O(\lambda)$ , respectively, and only implicitly gives the system characteristics. Canonical and condensed forms for generalized matrix pairs appearing in descriptor systems [5, 43] are out of the scope of this paper.

Given a controllability pair  $(A, B)$  there exists a *feedback equivalent* (also known as  $\Gamma$ -equivalent or block similar) matrix pair  $(A_B, B_B)$  in *Brunovsky canonical form* (BCF) [4, 28, 31], such that

$$P \begin{bmatrix} A - \lambda I_n & B \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ R & Q^{-1} \end{bmatrix} = \begin{bmatrix} A_B - \lambda I_n & B_B \end{bmatrix} = \left[ \begin{array}{cc|c} A_\epsilon & 0 & B_\epsilon \\ 0 & A_\mu & 0 \end{array} \right], \quad (3.2)$$

where  $A_\epsilon = \text{diag}(J_{\epsilon_1}(0), \dots, J_{\epsilon_{r_1}}(0))$ ,  $A_\mu = \text{diag}(J(\mu_1), \dots, J(\mu_q))$ , and  $B_\epsilon = \text{diag}(e_{\epsilon_1}, \dots, e_{\epsilon_{r_0}})$ . The transformation matrices  $P \in \mathbb{C}^{n \times n}$  and  $Q \in \mathbb{C}^{m \times m}$  are non-

singular and  $R \in \mathbb{C}^{m \times n}$ . Each block  $J(\mu_i)$  in  $A_\mu$  is block diagonal with the Jordan blocks for the specified finite eigenvalue  $\mu_i$ .  $J_{\epsilon_i}(0)$  is a nilpotent matrix in its reduced Jordan form and  $e_i = [0, \dots, 0, 1]^T \in \mathbb{C}^{i \times 1}$ . Moreover, the matrix pair  $(A_\epsilon, B_\epsilon)$  is controllable and corresponds to the  $L$  blocks in the KCF of  $\mathbf{S}_C(\lambda)$ . If  $\text{rank}(\mathbf{S}_C(\lambda)) < n$  for some  $\lambda \in \mathbb{C}$  then  $(A, B)$  is uncontrollable and there exists a regular pencil  $A_\mu$  whose eigenvalues correspond to the *uncontrollable eigenvalues (modes)*.

The dual form of BCF for the observability pair  $(A, C)$  is

$$\begin{bmatrix} P & S \\ 0 & T \end{bmatrix} \begin{bmatrix} A - \lambda I_n \\ C \end{bmatrix} P^{-1} = \begin{bmatrix} A_B - \lambda I_n \\ C_B \end{bmatrix} = \begin{bmatrix} A_\eta & 0 \\ -\frac{0}{C_\eta} & -\frac{A_\mu}{0} \end{bmatrix}, \quad (3.3)$$

where  $A_\eta = \text{diag}(J_{\eta_1}(0), \dots, J_{\eta_1}(0))$ ,  $A_\mu = \text{diag}(J(\mu_1), \dots, J(\mu_q))$ , and  $C_\eta = \text{diag}(e_{\eta_1}^T, \dots, e_{\eta_0}^T)$ . The transformation matrices  $P \in \mathbb{C}^{n \times n}$  and  $T \in \mathbb{C}^{p \times p}$  are non-singular and  $S \in \mathbb{C}^{n \times p}$ . The matrix pair  $(A_\eta, C_\eta)$  is observable and corresponds to the  $L^T$  blocks. If  $\text{rank}(\mathbf{S}_O(\lambda)) < n$  for some  $\lambda \in \mathbb{C}$  then  $(A, C)$  is unobservable and there exists a regular pencil  $A_\mu$  whose eigenvalues correspond to the *unobservable eigenvalues (modes)*.

Some of the system characteristics that the BCF directly reveals are:  $(A, B)$  has exactly  $m$   $L$  blocks, one for each column in  $B_\epsilon$ , and  $m - \text{rank}(B_B)$   $L_0$  blocks. Likewise,  $(A, C)$  has exactly  $p$   $L^T$  blocks, one for each row in  $C_\eta$ , and  $p - \text{rank}(C_B)$   $L_0^T$  blocks. Since  $\epsilon_{r_1+1} = \dots = \epsilon_{r_0} = 0$ , the column vectors  $e_{\epsilon_{r_1+1}}, \dots, e_{\epsilon_{r_0}}$  are  $0 \times 1$  and correspond to the  $L_0$  blocks;  $\text{rank}(B) = m - \#(L_0 \text{ blocks})$ . For each  $L_0$  block one input signal  $u_k(t)$  can be removed without losing controllability of  $(A_\epsilon, B_\epsilon)$ . Likewise, the row vectors  $e_{\eta_1+1}^T, \dots, e_{\eta_0}^T$  are  $1 \times 0$  and correspond to the  $L_0^T$  blocks, where for each  $L_0^T$  block one output signal  $y_k(t)$  can be removed without losing observability of  $(A_\eta, C_\eta)$ .

**4. The system pencil space.** An  $n \times (n + m)$  controllability pair  $(A, B)$  has  $n^2 + nm$  free elements and therefore belongs to an  $(n^2 + nm)$ -dimensional (*system pencil*) space, one dimension for each parameter. A controllability pair  $(A, B)$  can be seen as a point in the  $(n^2 + nm)$ -dimensional space, and the union of equivalent matrix pairs as a manifold in this space [17, 18]. Similarly, the  $(n + p) \times n$  observability pair  $(A, C)$  is a point in an  $(n^2 + np)$ -dimensional system pencil space. We say that the matrix pair “lives” in the space spanned by the manifold, and the dimension of the manifold is given from the number of parameters of the matrix pair, where each fixed parameter gives one less degree of freedom. The dimension of the complementary space to the manifold is called the *codimension*.

The *orbit* of a matrix pair,  $\mathcal{O}(A, B)$  or  $\mathcal{O}(A, C)$ , is a manifold of all equivalent matrix pairs, i.e., manifolds in the  $(n^2 + nm)$ -dimensional and  $(n^2 + np)$ -dimensional spaces, respectively. In the following, when something holds for both  $(A, B)$  and  $(A, C)$  we denote the matrix pairs with  $(*)$ , e.g.,  $\mathcal{O}(*)$ . Throughout this paper we only consider orbits under feedback equivalence [4, 31], which for the controllability pairs is defined as

$$\mathcal{O}(A, B) = \left\{ P \begin{bmatrix} A - \lambda I & B \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ R & Q^{-1} \end{bmatrix} : \det(P) \cdot \det(Q) \neq 0 \right\},$$

and for observability pairs as

$$\mathcal{O}(A, C) = \left\{ \begin{bmatrix} P & S \\ 0 & T \end{bmatrix} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} P^{-1} : \det(P) \cdot \det(T) \neq 0 \right\}.$$

In other words, all matrix pairs in the same orbit have the same canonical form, with the eigenvalues and the sizes of the Jordan blocks fixed. A *bundle* defines the union of all orbits with the same canonical form but with the eigenvalues unspecified,  $\bigcup_{\mu_i} \mathcal{O}(\ast)$  [1]. We denote the bundle of a matrix pair by  $\mathcal{B}(\ast)$ .

The dimension of the space  $\mathcal{O}(A, B)$  is equal to the dimension of the *tangent space* to  $\mathcal{O}(A, B)$  at  $(A, B)$ , denoted by  $\tan(A, B)$ . Similar definitions hold for the matrix pair  $(A, C)$ . The tangent spaces  $\tan(A, B)$  and  $\tan(A, C)$  can be represented in matrix form as

$$\begin{bmatrix} T_A & T_B \end{bmatrix} = X \begin{bmatrix} A & B \end{bmatrix} + \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} -X & 0 \\ V & W \end{bmatrix},$$

and

$$\begin{bmatrix} T_A \\ T_C \end{bmatrix} = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} + \begin{bmatrix} A \\ C \end{bmatrix} \begin{bmatrix} -X \end{bmatrix},$$

respectively, where  $X, Y, Z, V$  and  $W$  are matrices of conforming sizes [7].

Using the technique in [17], the tangent vectors  $\begin{bmatrix} T_A & T_B \end{bmatrix}$  can be expressed in terms of the  $\text{vec}$ -operator and Kronecker products (see also [7]):

$$\begin{bmatrix} \text{vec}(T_A) \\ \text{vec}(T_B) \end{bmatrix} = T_{(A,B)} \begin{bmatrix} \text{vec}(X) \\ \text{vec}(V) \\ \text{vec}(W) \end{bmatrix},$$

where  $\tan(A, B)$  is the range of the  $(n^2 + nm) \times (n^2 + nm + m^2)$  matrix

$$T_{(A,B)} = \begin{bmatrix} A^T \otimes I_n - I_n \otimes A & I_n \otimes B & 0 \\ B^T \otimes I_n & 0 & I_m \otimes B \end{bmatrix}. \quad (4.1)$$

Similarly,  $\tan(A, C)$  is the range of the  $(n^2 + np) \times (n^2 + np + p^2)$  matrix

$$T_{(A,C)} = \begin{bmatrix} A^T \otimes I_n - I_n \otimes A & C^T \otimes I_n & 0 \\ -I_n \otimes C & 0 & C^T \otimes I_p \end{bmatrix}, \text{ where} \quad (4.2)$$

$$\begin{bmatrix} \text{vec}(T_A) \\ \text{vec}(T_C) \end{bmatrix} = T_{(A,C)} \begin{bmatrix} \text{vec}(X) \\ \text{vec}(Y) \\ \text{vec}(Z) \end{bmatrix}.$$

The orthogonal complement of the tangent space is the *normal space*,  $\text{nor}(\ast)$ . The dimension of the normal space is called the *codimension* of  $\mathcal{O}(\ast)$  [12, 52], denoted by  $\text{cod}(\ast)$ . Together, the tangent and the normal spaces span the complete  $(n^2 + nm)$ -dimensional space for  $(A, B)$  and the complete  $(n^2 + np)$ -dimensional space for  $(A, C)$ .

Knowing the canonical structure, the explicit expression for the codimension of the controllability pair  $(A, B)$  is derived in [24], see also [25]. By rewriting the result, it is obvious that the computation of the codimension of  $(A, B)$  can be done using parts of the expression for matrix pencils [12]. The codimension of the observability pair  $(A, C)$  is easily derived by its duality to  $(A, B)$ . In summary, the codimension of the orbit of a controllability pair  $(A, B)$ , with the column minimal indices  $\epsilon_1, \dots, \epsilon_{r_0}$  and the finite elementary divisors  $h_1^{(i)}, \dots, h_{g_i}^{(i)}$  for each distinct eigenvalue  $\mu_i$ , is

$$\text{cod}(A, B) = c_{\text{Right}} + c_{\text{Jor}} + c_{\text{Jor,Right}}, \quad (4.3)$$



where

$$c_{\text{Right}} = \sum_{\epsilon_k > \epsilon_l} (\epsilon_k - \epsilon_l - 1), \quad c_{\text{Jor}} = \sum_{i=1}^q \sum_{k=1}^{g_i} (2k-1)h_k^{(i)}, \quad \text{and} \quad c_{\text{Jor,Right}} = r_0 \sum_{i=1}^q \sum_{k=1}^{g_i} h_k^{(i)}.$$

The codimension of the orbit of an observability pair  $(A, C)$ , with the row minimal indices  $\eta_1, \dots, \eta_{l_0}$  and the finite elementary divisors  $h_1^{(i)}, \dots, h_{g_i}^{(i)}$  for each distinct eigenvalue  $\mu_i$ , is

$$\text{cod}(A, C) = c_{\text{Left}} + c_{\text{Jor}} + c_{\text{Jor,Left}}, \quad (4.4)$$

where

$$c_{\text{Left}} = \sum_{\eta_k > \eta_l} (\eta_k - \eta_l - 1), \quad c_{\text{Jor}} = \sum_{i=1}^q \sum_{k=1}^{g_i} (2k-1)h_k^{(i)}, \quad \text{and} \quad c_{\text{Jor,Left}} = l_0 \sum_{i=1}^q \sum_{k=1}^{g_i} h_k^{(i)}.$$

The value of the eigenvalues make no contribution to the codimension in the bundle case. Therefore, knowing the codimension of an orbit the codimension of the corresponding bundle is one less for each distinct eigenvalue:  $\text{cod}(\mathcal{B}(*)) = \text{cod}(\mathcal{O}(*)) - (\text{number of distinct eigenvalues})$ . For example, if we are interested in a matrix pair  $(A, B)$  with  $k$  unspecified eigenvalues and the rest with known specified values, the codimension of  $\mathcal{B}(A, B)$  is  $\text{cod}(\mathcal{O}(A, B)) - k$ .

**5. Stratification of orbits and bundles.** In this section, we present the stratification of orbits and bundles of matrix pairs  $(A, B)$  and  $(A, C)$ . The most and least generic cases are considered in Section 5.1, and in Section 5.2 the coin rules representing the closure and cover relations are derived.

A stratification is a closure hierarchy of orbits (or bundles). Following [23, 18], we represent the stratification by a connected graph where the nodes correspond to orbits (or bundles) of canonical structures and the edges to their covering relations, see Figures 1.1 and 6.2. The graph is organized from top to bottom with nodes in increasing order of codimension.

Given a node representing an orbit (or bundle) of a canonical structure, the *closure* of that orbit (or bundle) includes the orbit (or bundle) itself and all orbits (or bundles) represented by the nodes which can be reached by a *downward path*. A downward path is defined as a path for which all edges start in a node and end in another node below in the graph. An *upward path* is a path in the opposite direction. In the following, when it is clear from context we use the shorter term structure when we refer to a canonical structure.

Given a matrix pair and its corresponding node in the graph, it is always possible to make the pair more generic by a small perturbation, i.e., change the pair to one corresponding to a node along an upward path from the node. It is normally not possible to make a corresponding downward move by a small perturbation, i.e., a structure is not, in general, near any of the more degenerate structures below in the graph. However, the cases when a structure below in the hierarchy actually is nearby is often of particular interest, as it shows that a more degenerate structure can be found by a small perturbation.

**5.1. Most and least generic cases.** Almost all matrix pairs of the same size and type (controllability or observability pairs) have the same canonical structure. This canonical structure corresponds to the most generic case and has the lowest

codimension in the closure hierarchy. The opposite case is the least generic case, or equivalently, the most degenerate case with the highest codimension. In the closure hierarchy graph, the most generic case is represented by the topmost node and the most degenerate case by the bottom node. The canonical structures in between correspond to degenerate (or non-generic) cases, which from a computational point of view can be a real challenge [14, 15].

The most generic structure of the controllability pair  $(A, B)$  has  $\mathcal{R} = (r_0, \dots, r_\alpha, r_{\alpha+1})$  where  $r_0 = \dots = r_\alpha = m$ ,  $r_{\alpha+1} = n \bmod m$ , and  $\alpha = \lfloor n/m \rfloor$  [29, 53]. For the observability pair  $(A, C)$  the most generic structure has  $\mathcal{L} = (l_0, \dots, l_\alpha, l_{\alpha+1})$  where  $l_0 = \dots = l_\alpha = p$ ,  $l_{\alpha+1} = n \bmod p$ , and  $\alpha = \lfloor n/p \rfloor$ . The most degenerate controllability pair has  $m$   $L_0$  blocks and  $n$  Jordan blocks of size  $1 \times 1$  corresponding to an eigenvalue of multiplicity  $n$ . Similarly, the most degenerate observability pair has  $p$   $L_0^T$  blocks and  $n$   $1 \times 1$  Jordan blocks. In other words, the most generic cases of the matrix pairs correspond to completely controllable and observable systems, while the most degenerate cases correspond to systems with  $n$  uncontrollable and  $n$  unobservable multiple modes, respectively.

We remark that the above formulae to compute the most generic structure only hold if there are no restrictions on the matrix pair. Otherwise, for example when the matrix pair has a special structure or fixed rank, the restrictions must be considered when determining the most and least generic cases. There can even exist several most generic structures, but only one with codimension 0 (if it exists). This has recently been studied for general matrix pencils in, e.g., [9, 10, 37].

**5.2. Closure and cover relations.** To determine the closure hierarchy for  $n \times (n + m)$  controllability pairs we stratify the  $(n^2 + nm)$ -dimensional system pencil space into feedback equivalent orbits (or bundles). Similarly, the closure hierarchy for  $(n + p) \times n$  observability pairs is determined by the stratification of feedback equivalent orbits (or bundles) in the  $(n^2 + np)$ -dimensional system pencil space. The stratification of orbits or bundles is given from the closure relations and further the cover relations between these manifolds, see Arnold [1] and [17, 18]. An orbit *covers* another orbit if its closure includes the closure of the other orbit and there is no orbit in between in the closure hierarchy, i.e., they are nearest neighbours in the hierarchy. The closure and cover relations for bundles are defined analogously.

Before we give the closure and cover relations for matrix pairs, we review some results for matrices and general matrix pencils.

From the closure condition for nilpotent matrices derived in [1, 18] and the definition of covering partitions, the cover relations for orbits of nilpotent matrices are obtained [18]. The orbit of a matrix is the manifold of all similar matrices:  $\mathcal{O}(A) = \{PAP^{-1} : \det(A) \neq 0\}$ . If the matrix  $A$  has well clustered eigenvalues but is not nilpotent, we order the Jordan blocks such that  $A = \text{diag}(A_1, \dots, A_q)$ , where  $A_i$  contains all Jordan blocks associated with the eigenvalue  $\mu_i$ . Then for each matrix  $A_i$ , we consider  $\tilde{A}_i = A_i - \mu_i I$  which is nilpotent, and the closure and cover relations for nilpotent matrices are applicable. It follows that the number of eigenvalues and the total size of all blocks associated with the same eigenvalue, are the same for all orbits in the closure hierarchy. This is in contrast to the bundle case where eigenvalues can coalesce or split apart.

**THEOREM 5.1. [1, 18]**  $\mathcal{O}(A_1)$  *covers*  $\mathcal{O}(A_2)$  if and only if *some*  $\mathcal{J}_{\mu_i}(A_2)$  *can be obtained from*  $\mathcal{J}_{\mu_i}(A_1)$  *by a minimum leftward coin move, and*  $\mathcal{J}_{\mu_j}(A_2) = \mathcal{J}_{\mu_j}(A_1)$  *for all*  $\mu_j \neq \mu_i$ .

In the case of not well-clustered eigenvalues, we have to consider the bundle case as

defined by Arnold [1]. Even if testing for closure relations between nilpotent matrices is trivial, deciding if one bundle is in the closure of another bundle is an NP-complete problem [18, 32]. The solution to the closure decision problem for matrix bundles is given in [16, 18, 45], and the cover relations expressed in terms of coin moves in [18].

The necessary conditions for an orbit or a bundle of two matrix pencils to be closest neighbours in a closure hierarchy were derived in [3, 8, 50], where the orbit is the manifold of strictly equivalent matrix pencils:  $\mathcal{O}(G - \lambda H) = \{U(G - \lambda H)V^{-1} : \det(U) \cdot \det(V) \neq 0\}$ . These conditions were later complemented with the corresponding sufficient conditions in [18]. Notice that in the following theorem, for the structure integer partition  $\mathcal{J}_{\mu_i}$  the eigenvalue  $\mu_i$  belongs to the extended complex plane  $\overline{\mathbb{C}}$ , i.e.,  $\mu_i \in \mathbb{C} \cup \{\infty\}$ . Furthermore, the restrictions on  $r_0$  and  $l_0$  in rules 1 and 2 correspond to that the number of  $L_k$  and  $L_k^T$  blocks cannot change.

**THEOREM 5.2.** [18] *Given the structure integer partitions  $\mathcal{L}$ ,  $\mathcal{R}$  and  $\mathcal{J}_{\mu_i}$  of  $G - \lambda H$ , where  $\mu_i \in \overline{\mathbb{C}}$ , one of the following if-and-only-if rules finds  $\tilde{G} - \lambda \tilde{H}$  such that  $\mathcal{O}(G - \lambda H)$  covers  $\mathcal{O}(\tilde{G} - \lambda \tilde{H})$ :*

- (1) *Minimum rightward coin move in  $\mathcal{R}$  (or  $\mathcal{L}$ ).*
- (2) *If the rightmost column in  $\mathcal{R}$  (or  $\mathcal{L}$ ) is one single coin, move that coin to a new rightmost column of some  $\mathcal{J}_{\mu_i}$  (which may be empty initially).*
- (3) *Minimum leftward coin move in any  $\mathcal{J}_{\mu_i}$ .*
- (4) *Let  $k$  denote the total number of coins in all of the longest (= lowest) rows from all of the  $\mathcal{J}_{\mu_i}$ . Remove these  $k$  coins, add one more coin to the set, and distribute  $k + 1$  coins to  $r_p$ ,  $p = 0, \dots, t$  and  $l_q$ ,  $q = 0, \dots, k - t - 1$  such that at least all nonzero columns of  $\mathcal{R}$  and  $\mathcal{L}$  are given coins.*

*Rules 1 and 2 are not allowed to make coin moves that affect  $r_0$  (or  $l_0$ ).*

Necessary and sufficient conditions for closure relations between orbits of matrix pairs  $(A, B)$  have been studied in [31], and later in [35, 36]. These are a subset of those for general matrix pencils. Here we give our reformulation and slight modification of the theorem originally presented in [36, Theorem 4.6] for orbits and the corresponding theorem for bundles, where  $\overline{\mathcal{O}}$  denotes the orbit closure and  $\overline{\mathcal{B}}$  is the bundle closure.

**THEOREM 5.3.** [36, 42]  *$\overline{\mathcal{O}}(A, B) \supseteq \overline{\mathcal{O}}(\tilde{A}, \tilde{B})$  if and only if the following conditions hold:*

- (1)  $\mathcal{R}(A, B) \geq \mathcal{R}(\tilde{A}, \tilde{B})$ .
- (2)  $\mathcal{J}_{\mu_i}(A, B) \leq \mathcal{J}_{\mu_i}(\tilde{A}, \tilde{B})$ , for all  $\mu_i \in \mathbb{C}$ ,  $i = 1, \dots, q$ .

**THEOREM 5.4.** *If  $\mathcal{B}(A, B)$  has at least as many distinct eigenvalues as  $\mathcal{B}(\tilde{A}, \tilde{B})$ , then  $\overline{\mathcal{B}}(A, B) \supseteq \overline{\mathcal{B}}(\tilde{A}, \tilde{B})$  if and only if the following conditions hold:*

- (1)  $\mathcal{R}(A, B) \geq \mathcal{R}(\tilde{A}, \tilde{B})$ .
- (2) *It is possible to coalesce eigenvalues and apply the dominance ordering coin moves to  $\mathcal{J}_{\mu_i}(A, B)$ , for any  $\mu_i$ , to reach  $(\tilde{A}, \tilde{B})$ .*

*Proof.* The theorem follows directly from Theorem 5.3 and the closure condition for matrix bundles presented in [18].  $\square$

The conditions for closure relations between two observability matrix pairs  $(A, C)$  are, from the duality with  $(A, B)$ , equal to those for  $(A, B)$  except that  $\mathcal{R}$  is replaced by  $\mathcal{L}$ .

In [35], also the necessary conditions for cover relations of matrix pencils with no row minimal indices have been derived. A matrix pencil  $G - \lambda H$  with no row minimal indices differs from a controllability pair  $(A, B)$  in that it can have infinite elementary divisors, which is not the case for standard matrix pairs. The cover relations [35, Proposition 5.2] are summarized in Proposition 5.5 with some minor reformulations,

where the invariants of  $G - \lambda H$  and  $\tilde{G} - \lambda \tilde{H}$  are

$$\begin{aligned} \epsilon &= (\epsilon_1, \dots, \epsilon_{r_0}), \quad h_{\mu_i} = (h_1^{(i)}, \dots, h_{g_i}^{(i)}), \quad s = (s_1, \dots, s_{g_\infty}), \quad \text{and} \\ \tilde{\epsilon} &= (\tilde{\epsilon}_1, \dots, \tilde{\epsilon}_{\tilde{r}_0}), \quad \tilde{h}_{\mu_j} = (\tilde{h}_1^{(j)}, \dots, \tilde{h}_{\tilde{g}_j}^{(j)}), \quad \tilde{s} = (\tilde{s}_1, \dots, \tilde{s}_{\tilde{g}_\infty}), \end{aligned}$$

respectively. Remark, the integer partitions associated with the same invariants of  $G - \lambda H$  and  $\tilde{G} - \lambda \tilde{H}$ , e.g.  $\epsilon$  and  $\tilde{\epsilon}$ , can be of different length.

**PROPOSITION 5.5. [35]** *Let  $G - \lambda H$  and  $\tilde{G} - \lambda \tilde{H}$  be two  $n \times (n + m)$  matrix pencils with no row minimal indices. If  $\mathcal{O}(G - \lambda H)$  covers  $\mathcal{O}(\tilde{G} - \lambda \tilde{H})$  then one of the following conditions holds:*

- (1)  $\text{conj}(\epsilon) > \text{conj}(\tilde{\epsilon})$  are adjacent,  $h_{\mu_i} = \tilde{h}_{\mu_i}$  for all eigenvalues  $\mu_i$ , and  $s = \tilde{s}$ .
- (2)  $\sum_{i=1}^m \epsilon_i > \sum_{i=1}^m \tilde{\epsilon}_i$ ,  $\text{conj}(\epsilon) > \text{conj}(\tilde{\epsilon})$  are adjacent,  $\tilde{h}_1^{(i)} = h_1^{(i)} + 1$  for some eigenvalue  $\mu_i$  (where  $\mu_i$  can be a new eigenvalue), and  $s = \tilde{s}$ .
- (3)  $\sum_{i=1}^m \epsilon_i > \sum_{i=1}^m \tilde{\epsilon}_i$ ,  $\text{conj}(\epsilon) > \text{conj}(\tilde{\epsilon})$  are adjacent,  $h_{\mu_i} = \tilde{h}_{\mu_i}$  for all eigenvalues  $\mu_i$ , and  $\tilde{s}_1 = s_1 + 1$  (where  $s$  and  $\tilde{s}$  can be empty partitions).
- (4)  $\epsilon = \tilde{\epsilon}$ ,  $h_{\mu_i} > \tilde{h}_{\mu_i}$  for all eigenvalues  $\mu_i$ , and  $s = \tilde{s}$ .
- (5)  $\epsilon = \tilde{\epsilon}$ ,  $h_{\mu_i} = \tilde{h}_{\mu_i}$  for all eigenvalues  $\mu_i$ , and  $s > \tilde{s}$ .

From Theorem 5.3, Proposition 5.5, and the cover conditions for matrix pencils in Theorem 5.2, it is possible to derive both necessary and sufficient conditions for a covering relation between two controllability pairs  $(A, B)$ . The result is given in Theorem 5.6, where  $r_0(A, B)$  denotes the number of column minimal indices for  $(A, B)$ . The proof is organized as follows. We modify Proposition 5.5 so that it fulfills the restrictions given by the structure of the controllability pair and then, where required, strengthen each condition so that they become not only necessary but also sufficient.

**THEOREM 5.6.**  $\mathcal{O}(A, B)$  covers  $\mathcal{O}(\tilde{A}, \tilde{B})$  if and only if one of the following conditions holds:

- (1)  $\mathcal{R}(A, B)$  covers  $\mathcal{R}(\tilde{A}, \tilde{B})$  where  $r_0(A, B) = r_0(\tilde{A}, \tilde{B})$ , and  $\mathcal{J}_{\mu_i}(A, B) = \mathcal{J}_{\mu_i}(\tilde{A}, \tilde{B})$  for all eigenvalues  $\mu_i$ .
- (2) If  $r_{\epsilon_1} = 1$  and  $\epsilon_1 \geq 1$  for  $\mathcal{R}(A, B)$ , then  $\mathcal{R}(\tilde{A}, \tilde{B}) = \mathcal{R}(A, B) \setminus (r_{\epsilon_1})$ ,  $\mathcal{J}_{\mu_i}(\tilde{A}, \tilde{B}) = \mathcal{J}_{\mu_i}(A, B) \cup (1)$  for some eigenvalue  $\mu_i$  (where  $\mathcal{J}_{\mu_i}(A, B)$  can be an empty partition), and  $\mathcal{J}_{\mu_j}(A, B) = \mathcal{J}_{\mu_j}(\tilde{A}, \tilde{B})$  for all  $\mu_j \neq \mu_i$ .
- (3)  $\mathcal{R}(A, B) = \mathcal{R}(\tilde{A}, \tilde{B})$ ,  $\mathcal{J}_{\mu_i}(A, B)$  covers  $\mathcal{J}_{\mu_i}(\tilde{A}, \tilde{B})$  for one eigenvalue  $\mu_i$ , and  $\mathcal{J}_{\mu_j}(A, B) = \mathcal{J}_{\mu_j}(\tilde{A}, \tilde{B})$  for all  $\mu_j \neq \mu_i$ .

*Proof.* Let 5.5(n) denote condition  $n$  of Proposition 5.5, and similarly, 5.6(m) denotes condition  $m$  of Theorem 5.6.

A matrix pencil  $G - \lambda H$  with no row minimal indices can have infinite elementary divisors which a controllability pair  $(A, B)$  cannot have. This restriction is introduced by only considering finite elementary divisors, which obviously exclude 5.5(3) and 5.5(5) (where  $G - \lambda H$  and/or  $\tilde{G} - \lambda \tilde{H}$  have infinite elementary divisors). The remaining three conditions are now considered, and we begin each proof by rewriting the conditions in the structure integer notation:  $\mathcal{R}$ ,  $\mathcal{L}$ , and  $\mathcal{J}$ .

First we consider 5.5(1) which can be rewritten as:

$$\mathcal{R}(A, B) > \mathcal{R}(\tilde{A}, \tilde{B}) \text{ are adjacent and } \mathcal{J}_{\mu_i}(A, B) = \mathcal{J}_{\mu_i}(\tilde{A}, \tilde{B}).$$

Since the two matrix pairs have the same Jordan structure, the size of the right singular parts of  $(A, B)$  and  $(\tilde{A}, \tilde{B})$  must be equal, i.e.,  $\sum \mathcal{R}(A, B) = \sum \mathcal{R}(\tilde{A}, \tilde{B})$ . Consequently,  $\mathcal{R}(A, B) > \mathcal{R}(\tilde{A}, \tilde{B})$  are adjacent is strengthened to  $\mathcal{R}(A, B)$  covers  $\mathcal{R}(\tilde{A}, \tilde{B})$ . This is also remarked in [35, proof of Theorem 5.1]. A consequence of the change of representation from column minimal indices to  $\mathcal{R}$ , is that we in 5.6(1) have

to introduce the restriction that  $r_0$  may not be affected. Otherwise the number of column minimal indices may change. The new condition is given in 5.6(1).

Now consider 5.5(2) which can be rewritten as:

$$\sum \mathcal{R}(A, B) > \sum \mathcal{R}(\tilde{A}, \tilde{B}), \mathcal{R}(A, B) > \mathcal{R}(\tilde{A}, \tilde{B}) \text{ are adjacent, and} \\ \mathcal{J}_{\mu_i}(\tilde{A}, \tilde{B}) = \mathcal{J}_{\mu_i}(A, B) \cup (1) \text{ for some } \mu_i \text{ (where } \mu_i \text{ can be a new} \\ \text{eigenvalue)}.$$

If  $\sum \mathcal{R}(A, B) > \sum \mathcal{R}(\tilde{A}, \tilde{B})$  then  $\mathcal{R}(A, B) > \mathcal{R}(\tilde{A}, \tilde{B})$  are adjacent if and only if  $\mathcal{R}(\tilde{A}, \tilde{B})$  can be derived from  $\mathcal{R}(A, B)$  in the following way. If  $r_{\epsilon_1} = 1$  and  $\epsilon_1 \geq 1$  for  $\mathcal{R}(A, B)$ , then  $\mathcal{R}(\tilde{A}, \tilde{B}) = \mathcal{R}(A, B) \setminus (r_{\epsilon_1})$  [11]. Furthermore, the regular part is expanded by increasing the largest block for some eigenvalue by one, or by creating a  $1 \times 1$  block for a new eigenvalue. It follows that condition 5.5(2) corresponds to rule (2) for orbits of matrix pencils, which already fulfills both the necessary and sufficient conditions, and we have 5.6(2).

Finally, 5.5(4) can be rewritten as:

$$\mathcal{R}(A, B) = \mathcal{R}(\tilde{A}, \tilde{B}) \text{ and } \mathcal{J}_{\mu_i}(A, B) < \mathcal{J}_{\mu_i}(\tilde{A}, \tilde{B}) \text{ for all } \mu_i.$$

This condition considers the case when the two matrix pairs have equal right singular parts, as opposed to 5.5(1) where the regular parts are the same. The conditions  $\mathcal{R}(A, B) = \mathcal{R}(\tilde{A}, \tilde{B})$  and  $\mathcal{J}_{\mu_i}(A, B) < \mathcal{J}_{\mu_i}(\tilde{A}, \tilde{B})$  do not guarantee that  $(A, B)$  covers  $(\tilde{A}, \tilde{B})$ . To guarantee that  $(A, B)$  covers  $(\tilde{A}, \tilde{B})$  the corresponding integer partitions  $\mathcal{J}_{\mu_i}(A, B)$  and  $\mathcal{J}_{\mu_i}(\tilde{A}, \tilde{B})$  must also cover each other, which corresponds to the matrix case (Theorem 5.1). The new condition is given in 5.6(3).  $\square$

**THEOREM 5.7.**  $\mathcal{B}(A, B)$  covers  $\mathcal{B}(\tilde{A}, \tilde{B})$  if and only if one of the following conditions holds:

- (1)  $\mathcal{R}(A, B)$  covers  $\mathcal{R}(\tilde{A}, \tilde{B})$  where  $r_0(A, B) = r_0(\tilde{A}, \tilde{B})$ , and  $\mathcal{J}_{\mu_i}(A, B) = \mathcal{J}_{\mu_i}(\tilde{A}, \tilde{B})$  for all eigenvalues  $\mu_i$ .
- (2) If  $r_{\epsilon_1} = 1$  and  $\epsilon_1 \geq 1$  for  $\mathcal{R}(A, B)$ , then  $\mathcal{R}(\tilde{A}, \tilde{B}) = \mathcal{R}(A, B) \setminus (r_{\epsilon_1})$ ,  $\mathcal{J}_{\mu_i}(\tilde{A}, \tilde{B}) = (1)$  for a new eigenvalue  $\mu_i$ , and  $\mathcal{J}_{\mu_j}(A, B) = \mathcal{J}_{\mu_j}(\tilde{A}, \tilde{B})$  for all  $\mu_j \neq \mu_i$ .
- (3)  $\mathcal{R}(A, B) = \mathcal{R}(\tilde{A}, \tilde{B})$ ,  $\mathcal{J}_{\mu_i}(A, B)$  covers  $\mathcal{J}_{\mu_i}(\tilde{A}, \tilde{B})$  for one eigenvalue  $\mu_i$ , and  $\mathcal{J}_{\mu_j}(A, B) = \mathcal{J}_{\mu_j}(\tilde{A}, \tilde{B})$  for all  $\mu_j \neq \mu_i$ .
- (4)  $\mathcal{R}(A, B) = \mathcal{R}(\tilde{A}, \tilde{B})$ ,  $\mathcal{J}_{\mu_i}(\tilde{A}, \tilde{B}) = \mathcal{J}_{\mu_i}(A, B) \cup \mathcal{J}_{\mu_j}(A, B)$  for one pair of eigenvalues  $\mu_i$  and  $\mu_j$ ,  $\mu_i \neq \mu_j$ , and  $\mathcal{J}_{\mu_k}(A, B) = \mathcal{J}_{\mu_k}(\tilde{A}, \tilde{B})$  for all  $\mu_k \neq \mu_i, \mu_j$ .

*Proof.* The proof of the bundle case follows directly from Theorem 5.6 and the covering rules for bundles of matrix pencils given in [18].  $\square$

Notably, Theorem 5.7 has four rules in contrary to Theorem 5.6 which has three rules. The additional rule (4) follows from that eigenvalues can coalesce in the bundle case.

From the dual relation between the controllability pair  $(A, B)$  and the observability pair  $(A, C)$ , it follows that replacing partition  $\mathcal{R}$  by  $\mathcal{L}$  in Theorems 5.6 and 5.7 give the cover conditions for the observability pair  $(A, C)$ . We remark that the theorems are only valid for independent matrix pairs  $(A, B)$  and  $(A, C)$ , respectively. They cannot be applied straightforwardly to the related matrix triple  $(A, B, C)$  or matrix quadruple  $(A, B, C, D)$ . The covering relations for orbits and bundles of the controllability and observability pairs in terms of coin rules are given in Corollaries 5.8 and 5.9. The reformulations are done using the definition of integer partitions in Section 2.2.

TABLE 5.1

Given the structure integer partitions  $\mathcal{R}$ ,  $\mathcal{L}$ , and  $\mathcal{J}_{\mu_i}$  of a matrix pair, one of the following if-and-only-if rules finds  $(\tilde{A}, \tilde{B})$  or  $(\tilde{A}, \tilde{C})$  fulfilling orbit or bundle covering relations with  $(A, B)$  or  $(A, C)$ , respectively.

<p><b>A.</b> <math>\mathcal{O}(A, B)</math> covers <math>\mathcal{O}(\tilde{A}, \tilde{B})</math>:</p> <ol style="list-style-type: none"> <li>(1) Minimum rightward coin move in <math>\mathcal{R}</math>.</li> <li>(2) If the rightmost column in <math>\mathcal{R}</math> is one single coin, move that coin to a new rightmost column of some <math>\mathcal{J}_{\mu_i}</math> (which may be empty initially).</li> <li>(3) Minimum leftward coin move in any <math>\mathcal{J}_{\mu_i}</math>.</li> </ol> <p>Rules 1 and 2 are not allowed to do coin moves that affect <math>r_0</math>.</p> <p><b>B.</b> <math>\mathcal{B}(A, B)</math> covers <math>\mathcal{B}(\tilde{A}, \tilde{B})</math></p> <ol style="list-style-type: none"> <li>(1) Same as rule <b>A</b>(1).</li> <li>(2) Same as rule <b>A</b>(2), except it is only allowed to start a new set corresponding to a new eigenvalue (i.e., no appending to nonempty sets).</li> <li>(3) Same as rule <b>A</b>(3).</li> <li>(4) Let any pair of eigenvalues coalesce, i.e., take the union of their sets of coins.</li> </ol>	<p><b>E.</b> <math>\mathcal{O}(A, C)</math> covers <math>\mathcal{O}(\tilde{A}, \tilde{C})</math>:</p> <ol style="list-style-type: none"> <li>(1) Minimum rightward coin move in <math>\mathcal{L}</math>.</li> <li>(2) If the rightmost column in <math>\mathcal{L}</math> is one single coin, move that coin to a new rightmost column of some <math>\mathcal{J}_{\mu_i}</math> (which may be empty initially).</li> <li>(3) Minimum leftward coin move in any <math>\mathcal{J}_{\mu_i}</math>.</li> </ol> <p>Rules 1 and 2 are not allowed to do coin moves that affect <math>l_0</math>.</p> <p><b>F.</b> <math>\mathcal{B}(A, C)</math> covers <math>\mathcal{B}(\tilde{A}, \tilde{C})</math>:</p> <ol style="list-style-type: none"> <li>(1) Same as rule <b>E</b>(1).</li> <li>(2) Same as rule <b>E</b>(2), except it is only allowed to start a new set corresponding to a new eigenvalue (i.e., no appending to nonempty sets).</li> <li>(3) Same as rule <b>E</b>(3).</li> <li>(4) Let any pair of eigenvalues coalesce, i.e., take the union of their sets of coins.</li> </ol>
<p><b>C.</b> <math>\mathcal{O}(A, B)</math> is covered by <math>\mathcal{O}(\tilde{A}, \tilde{B})</math></p> <ol style="list-style-type: none"> <li>(1) Minimum leftward coin move in <math>\mathcal{R}</math>, without affecting <math>r_0</math>.</li> <li>(2) If the rightmost column in some <math>\mathcal{J}_{\mu_i}</math> consists of one coin only, move that coin to a new rightmost column in <math>\mathcal{R}</math>.</li> <li>(3) Minimum rightward coin move in any <math>\mathcal{J}_{\mu_i}</math>.</li> </ol> <p><b>D.</b> <math>\mathcal{B}(A, B)</math> is covered by <math>\mathcal{B}(\tilde{A}, \tilde{B})</math></p> <ol style="list-style-type: none"> <li>(1) Same as rule <b>C</b>(1).</li> <li>(2) Same as rule <b>C</b>(2), except that <math>\mathcal{J}_{\mu_i}</math> must consist of one coin only.</li> <li>(3) Same as rule <b>C</b>(3).</li> <li>(4) For any <math>\mathcal{J}_{\mu_i}</math>, divide the set of coins into two new sets so that their union is <math>\mathcal{J}_{\mu_i}</math>.</li> </ol>	<p><b>G.</b> <math>\mathcal{O}(A, C)</math> is covered by <math>\mathcal{O}(\tilde{A}, \tilde{C})</math>:</p> <ol style="list-style-type: none"> <li>(1) Minimum leftward coin move in <math>\mathcal{L}</math>, without affecting <math>l_0</math>.</li> <li>(2) If the rightmost column in some <math>\mathcal{J}_{\mu_i}</math> consists of one coin only, move that coin to a new rightmost column in <math>\mathcal{L}</math>.</li> <li>(3) Minimum rightward coin move in any <math>\mathcal{J}_{\mu_i}</math>.</li> </ol> <p><b>H.</b> <math>\mathcal{B}(A, C)</math> is covered by <math>\mathcal{B}(\tilde{A}, \tilde{C})</math>:</p> <ol style="list-style-type: none"> <li>(1) Same as rule <b>G</b>(1).</li> <li>(2) Same as rule <b>G</b>(2), except that <math>\mathcal{J}_{\mu_i}</math> must consist of one coin only.</li> <li>(3) Same as rule <b>G</b>(3).</li> <li>(4) For any <math>\mathcal{J}_{\mu_i}</math>, divide the set of coins into two new sets so that their union is <math>\mathcal{J}_{\mu_i}</math>.</li> </ol>

**COROLLARY 5.8.** Given the structure integer partitions  $\mathcal{R}$  and  $\mathcal{J}_{\mu_i}$  of  $(A, B)$ , one of the if-and-only-if rules of **A–D** in Table 5.1 finds  $(\tilde{A}, \tilde{B})$  fulfilling orbit or bundle covering relations with  $(A, B)$ .

**COROLLARY 5.9.** Given the structure integer partitions  $\mathcal{L}$  and  $\mathcal{J}_{\mu_i}$  of  $(A, C)$ , one of the if-and-only-if rules of **E–H** in Table 5.1 finds  $(\tilde{A}, \tilde{C})$  fulfilling orbit or bundle covering relations with  $(A, C)$ .

The major difference between the rules for matrix pencils and matrix pairs, is that rule (4) in Theorem 5.2 does not apply to matrix pairs, since there is only one type of singular blocks ( $L_i$  or  $L_j^T$ ) in each matrix pair type. Moreover, rules (1) and (2) of **A–D** in Table 5.1 only apply to the structure integer partition  $\mathcal{R}$  and rules (1) and (2) of **E–H** in Table 5.1 only apply to  $\mathcal{L}$ .

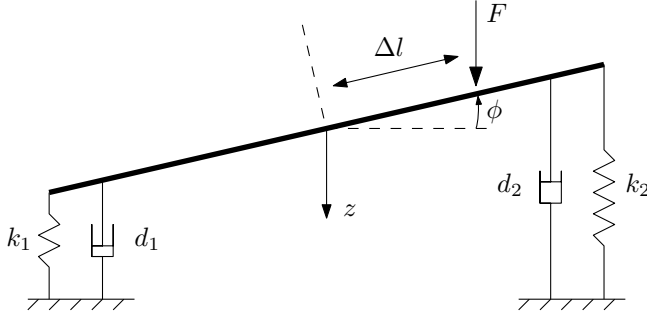


FIG. 6.1. Mechanical system consisting of a uniform platform controlled by a vertical force [44].

**6. Illustrating the stratification.** To illustrate the concept of stratification we consider two examples from systems and control applications. We use the software tool StratiGraph [38, 41] for computing and visualizing the closure hierarchy graphs for the different matrix pairs in the examples. The numerical results regarding Kronecker structure information and upper/lower bounds are computed using the prototype of the Matrix Canonical Structure (MCS) Toolbox for MATLAB [39, 22].

**6.1. Mechanical system.** The first example is a mechanical system studied by Mailybaev [44], see Figure 6.1. It consists of a thin uniform platform supported at both ends by springs, where the platform has mass  $m$  and length  $2l$ , and the springs have elasticity coefficients  $k_1, k_2$  and viscous damping coefficients  $d_1, d_2$ . The position of the platform is determined by the vertical coordinate  $z$  of its center and the angle  $\phi$  between the platform and the horizontal axis.

At distance  $\Delta l$ ,  $-1 \leq \Delta \leq 1$ , from the center of the platform a force  $F$  is applied, which is the control parameter of the system. The equilibrium of the system when  $F = 0$  is assumed to be  $z = 0$  and  $\phi = 0$ . For a zero force  $F$  and a nonzero  $z$  and/or  $\phi$ , the system oscillates with a decaying amplitude until it reaches equilibrium asymptotically. If the system is controllable, there exists a control action such that the system can be put into equilibrium in finite time. Otherwise, if it is uncontrollable or close to an uncontrollable system this task becomes difficult or even impossible.

By linearizing the equations of motion of the system near the equilibrium the system can be expressed by the state-space model  $\dot{x} = Ax(\tau) + Bu(\tau)$ , where the derivative is taken with respect to time  $\tau = t/\omega$  and  $\omega$  is a time scale coefficient. The resulting state-space model is

$$\begin{bmatrix} \omega \dot{z}/l \\ \omega \dot{\phi} \\ \omega^2 \ddot{z}/l \\ \omega^2 \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -c_1 & -c_2 & -f_1 & -f_2 \\ -3c_2 & -3c_1 & -3f_2 & -3f_1 \end{bmatrix} \begin{bmatrix} z/l \\ \phi \\ \omega \dot{z}/l \\ \omega \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3\Delta \end{bmatrix} \frac{\omega^2}{ml} F, \quad (6.1)$$

where

$$c_1 = \frac{(k_1 + k_2)\omega^2}{m}, \quad c_2 = \frac{(k_1 - k_2)\omega^2}{m}, \quad f_1 = \frac{(d_1 + d_2)\omega}{m}, \quad \text{and} \quad f_2 = \frac{(d_1 - d_2)\omega}{m}.$$

Let us consider a controllability pair of (6.1), denoted  $(A_0, B_0)$ , with the parameters  $d_1 = 4$ ,  $d_2 = 4$ ,  $k_1 = 6$ ,  $k_2 = 6$ ,  $m = 3$ ,  $l = 1$ ,  $\omega = 0.01$ , and  $\Delta = 0$ . The KCF of

$(A_0, B_0)$  is  $L_2 \oplus J_1(\alpha) \oplus J_1(\beta)$  with the corresponding Brunovsky canonical form

$$[A_B \quad B_B] - \lambda [I_4 \quad 0] = \left[ \begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 \end{array} \right] - \lambda \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right],$$

where  $\alpha = -0.02$  and  $\beta = -0.06$ . From the BCF of  $(A_0, B_0)$  we can directly see that the system is uncontrollable with the uncontrollable modes  $\alpha$  and  $\beta$ ;  $\text{rank}([A_B \quad B_B] - \lambda [I_4 \quad 0]) = 3$  for  $\lambda \in \{\alpha, \beta\}$ . The two uncontrollable modes correspond to that the angle  $\phi$  and its velocity  $\dot{\phi}$  cannot be controlled by the force  $F$ .

In [44], Mailybaev developed a quantitative perturbation method for local analysis of the uncontrollability set for a linear dynamical system depending on parameters. A *uncontrollability set* is defined as the set of values of a parameter vector  $p$  for which  $(A, B)$  depending on  $p$  is uncontrollable. In [44], an uncontrollable set for  $(A_0, B_0)$  is computed by letting the parameters  $c_1$  and  $f_1$  be fixed and varying the parameter vector  $p = (c_2, f_2, \Delta)$  in the range of  $-c_1 < c_2 < c_1$  and  $-f_1 < f_2 < f_1$ . It is also shown how the modes of  $(A_0, B_0)$  are changing over this set.

With the stratification theory, the quantitative results presented in [22, 44] and additional results like distance to uncontrollability [34, 46] are complemented with new qualitative information. In the following, we step-by-step illustrate the procedure to obtain the bundle stratification of the controllability pencil  $\mathbf{S}_C(\lambda)$  of size  $4 \times 5$ , which  $(A_0, B_0)$  is part of. Note that we can only change the values of the parameters  $c_1, c_2, f_1, f_2$ , and  $\Delta$  in the state-space model (6.1); The first two rows of the matrix  $A$  and the first three of  $B$  are fixed. As we will see, due to the special structure of  $A$  and  $B$  not all bundles or parts of these exist for  $(A, B)$ , which would exist for a controllability pair with unrestricted matrices  $A$  and  $B$ . We only show in details how to get the subgraph representing the stratification of possible structures. The complete bundle stratification of  $(A, B)$  is displayed in Figure 6.2, where the nodes corresponding to the bundles of possible structures are highlighted by the grey area. Let  $c:k$  denote node  $\binom{c}{k}$  in Figure 6.2, where  $c$  is the codimension of the corresponding bundle and  $k$  is an order number that identifies individual nodes with the same codimension.

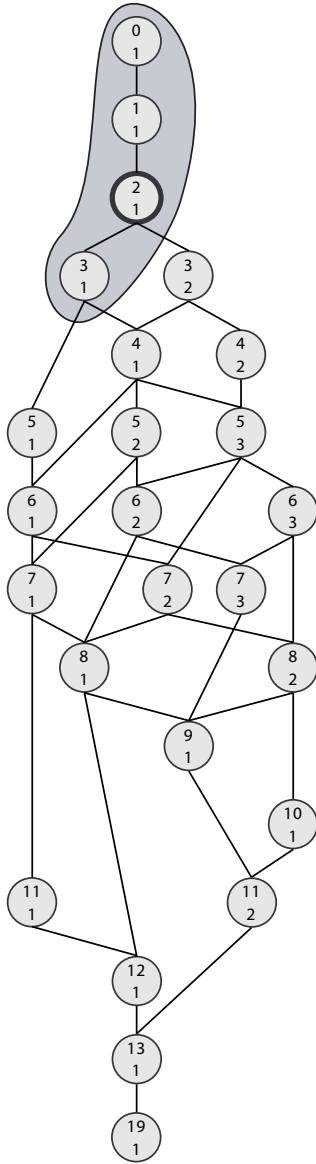
The first step is to compute the codimension of  $(A_0, B_0)$  using (4.3):  $\text{cod}(\mathcal{O}(A_0, B_0)) = 0 + (1 + 1) + 1(1 + 1) = 4$ . To get the codimension of the bundle the number of distinct eigenvalues are subtracted:  $\text{cod}(\mathcal{B}(A_0, B_0)) = 4 - 2 = 2$ . In Figure 6.2,  $\mathcal{B}(A_0, B_0)$  corresponds to node 2:1. To find covered or covering bundle(s) we use the set of rules **B** and **D**, respectively, in Table 5.1. To apply these rules we express the KCF of  $(A_0, B_0)$  in terms of its structure integer partitions:  $\mathcal{R} = (1, 1, 1)$ ,  $\mathcal{J}_\alpha = (1)$ , and  $\mathcal{J}_\beta = (1)$ . We are now ready to determine which bundle(s) that cover  $\mathcal{B}(A_0, B_0)$ .

Rule **D**(1) is not applicable because it would affect  $r_0$  (the first column of  $\mathcal{R}$ ). Rule **D**(2) can be applied to either  $\mathcal{J}_\alpha$  or  $\mathcal{J}_\beta$ , we choose the former:

$$\mathcal{R}: \circ \circ \circ, \quad \mathcal{J}_\alpha: \bullet, \quad \mathcal{J}_\beta: \circ \Rightarrow \mathcal{R}: \circ \circ \circ \bullet, \quad \mathcal{J}_\beta: \circ,$$

which gives the structure  $L_3 \oplus J_1(\beta)$ . The rules **D**(3) and **D**(4) are not applicable because  $\mathcal{J}_\alpha$  and  $\mathcal{J}_\beta$  only have one coin each. So the only bundle covering  $\mathcal{B}(A_0, B_0)$  is the bundle with KCF  $L_3 \oplus J_1(\beta)$ , which has codimension 1 and is represented by node 1:1 in Figure 6.2. Furthermore, this system is uncontrollable with one uncontrollable





**Codimension 0**

1  $L_4$

**Codimension 1**

1  $L_3 \oplus J_1(\mu_1)$

**Codimension 2**

1  $L_2 \oplus J_1(\mu_1) \oplus J_1(\mu_2)$

**Codimension 3**

1  $L_2 \oplus J_2(\mu_1)$

2  $L_1 \oplus J_1(\mu_1) \oplus J_1(\mu_2) \oplus J_1(\mu_3)$

**Codimension 4**

1  $L_1 \oplus J_2(\mu_1) \oplus J_1(\mu_2)$

2  $L_0 \oplus J_1(\mu_1) \oplus J_1(\mu_2) \oplus J_1(\mu_3) \oplus J_1(\mu_4)$

**Codimension 5**

1  $L_2 \oplus 2J_1(\mu_1)$

2  $L_1 \oplus J_3(\mu_1)$

3  $L_0 \oplus J_2(\mu_1) \oplus J_1(\mu_2) \oplus J_1(\mu_3)$

**Codimension 6**

1  $L_1 \oplus 2J_1(\mu_1) \oplus J_1(\mu_2)$

2  $L_0 \oplus J_3(\mu_1) \oplus J_1(\mu_2)$

3  $L_0 \oplus J_2(\mu_1) \oplus J_2(\mu_2)$

**Codimension 7**

1  $L_1 \oplus J_2(\mu_1) \oplus J_1(\mu_1)$

2  $L_0 \oplus 2J_1(\mu_1) \oplus J_1(\mu_2) \oplus J_1(\mu_3)$

3  $L_0 \oplus J_4(\mu_1)$

**Codimension 8**

1  $L_0 \oplus J_2(\mu_1) \oplus J_1(\mu_1) \oplus J_1(\mu_2)$

2  $L_0 \oplus 2J_1(\mu_1) \oplus J_2(\mu_2)$

**Codimension 9**

1  $L_0 \oplus J_3(\mu_1) \oplus J_1(\mu_1)$

**Codimension 10**

1  $L_0 \oplus 2J_1(\mu_1) \oplus 2J_1(\mu_2)$

**Codimension 11**

1  $L_1 \oplus 3J_1(\mu_1)$

2  $L_0 \oplus 2J_2(\mu_1)$

**Codimension 12**

1  $L_0 \oplus 3J_1(\mu_1) \oplus J_1(\mu_2)$

**Codimension 13**

1  $L_0 \oplus J_2(\mu_1) \oplus 2J_1(\mu_1)$

**Codimension 19**

1  $L_0 \oplus 4J_1(\mu_1)$

FIG. 6.2. The graph shows the complete bundle stratification of a  $4 \times 5$  controllability pencil  $\mathbf{S}_C(\lambda)$ , where the grey area marks the possible structures for the mechanical system (6.1). The upper number in each node is the codimension of the corresponding bundle. The lower number is an order number that identifies individual nodes with the same codimension. The table to the right of the graph displays the corresponding KCF structures associated with the nodes in the graph.

mode  $\beta = -0.06$ , which also can be seen from its BCF:

$$[A_B \quad B_B] - \lambda [I_4 \quad 0] = \begin{bmatrix} 0 & 1 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 1 \\ 0 & 0 & 0 & -0.06 & | & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & | & 0 \end{bmatrix}.$$

For the system (6.1), we can at least find two cases which belong to this bundle. The first one<sup>1</sup> occurs when the elasticity coefficients  $k_1$  and  $k_2$  are zero. This case is not of practical interest, since it corresponds to a system with no springs. The second case occurs when element  $A(4, 2) = 1.2e-3$  becomes zero and element  $A(4, 3)$  is perturbed with  $\epsilon \geq 1e-12$ . The KCF of this system is  $L_3 \oplus J_1(0)$ .

We continue by repeating the procedure for  $L_3 \oplus J_1(\beta)$ . As for the previous structure, the only rule applicable is **D**(2). So, we take the single coin in  $\mathcal{J}_\beta$  and move that to a new right-most column of  $\mathcal{R}$ :

$$\mathcal{R}: \circ \circ \circ \circ, \quad \mathcal{J}_\beta: \bullet \Rightarrow \mathcal{R}: \circ \circ \circ \circ \bullet,$$

which gives the KCF  $L_4$  with BCF:

$$[A_B \quad B_B] - \lambda [I_4 \quad 0] = \begin{bmatrix} 0 & 1 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & 0 & | & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & | & 0 \end{bmatrix}.$$

This is the most generic case represented by the topmost node 0:1 in Figure 6.2 and has codimension 0. As we can see from its BCF, it is controllable;  $\text{rank}([A_B \quad B_B] - \lambda [I_4 \quad 0]) = 4$  for all  $\lambda \in \mathbb{C}$ . In other words, there exists a control parameter  $F$  such that any state of  $z$  and  $\phi$  can be reached in finite time.

After having reached the most generic case and the top of the closure-hierarchy graph, we continue by determining the bundle(s) covered by  $\mathcal{B}(A_0, B_0)$  using the set of rules **B** in Table 5.1. But first, we remark that the mechanical system represented by the state-space system (6.1) must have an  $L$  block of at least size 2, i.e., it has at most two uncontrollable modes. This can be seen by studying the system with all parameters set to zero:

$$\begin{bmatrix} \omega \dot{z}/l \\ \omega \dot{\phi} \\ \omega^2 \ddot{z}/l \\ \omega^2 \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z/l \\ \phi \\ \omega \dot{z}/l \\ \omega \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \frac{\omega^2}{ml} F,$$

which has the KCF  $L_2 \oplus J_2(0)$ . The bundle of this canonical structure has codimension 3 and is represented by node 3:1 in Figure 6.2. Indeed, it is the most degenerate structure possible for the state-space system (6.1). As we can see from the graph in Figure 6.2,  $\mathcal{B}(L_2 \oplus J_2(0))$  is covered by  $\mathcal{B}(A_0, B_0)$ . This closure relation is obtained by applying rule **B**(4) to  $(A_0, B_0)$ :

$$\mathcal{R}: \circ \circ \circ, \quad \mathcal{J}_\alpha: \circ \cup \mathcal{J}_\beta: \bullet \Rightarrow \mathcal{R}: \circ \circ \circ, \quad \mathcal{J}_\alpha: \circ \bullet.$$

We can also reach this bundle by changing the value of  $m$  in  $(A_0, B_0)$ . Let  $(\tilde{A}_0, \tilde{B}_0)$  have the same parameters as  $(A_0, B_0)$  but with  $m$  unfixed. With  $m = 4$ ,  $(\tilde{A}_0, \tilde{B}_0)$  has KCF  $L_2 \oplus J_2(-0.2)$  and by a small perturbation on  $m$  we again reach the bundle of  $(A_0, B_0)$ ,  $\mathcal{B}(L_2 \oplus J_1(\mu_1) \oplus J_1(\mu_2))$ . Actually, for  $m < 4$   $(\tilde{A}_0, \tilde{B}_0)$  has KCF  $L_2 \oplus J_1(\mu_1) \oplus J_1(\mu_2)$  with two real eigenvalues, and for  $m > 4$  the system has instead one complex conjugate pair of eigenvalues.

The only other rule that can be applied to  $(A_0, B_0)$  is rule **B**(2), producing the structure  $L_1 \oplus J_1(\mu_1) \oplus J_1(\mu_2) \oplus J_1(\mu_3)$ . However, this structure has three uncontrollable modes which is not possible for the mechanical system considered. So, the closure

<sup>1</sup>The parameters in  $A$  are:  $c_1 = c_2 = 0$  and one of  $f_1$  and  $f_2$  is non-zero while the other one is equal to zero ( $\Delta$  is arbitrary).

hierarchy for the state-space system (6.1) corresponds to the highlighted subgraph of the complete bundle stratification of  $4 \times 5$  controllability pencil in Figure 6.2.

Notice, there also exists a structure  $L_2 \oplus 2J_1(\mu)$  (node 5:1) in the closure hierarchy which has an  $L$  block of size at least two, and therefore also should be possible. However, since the codimension of  $\mathcal{B}(L_2 \oplus 2J_1(\mu))$  is less than the most degenerate case  $L_2 \oplus J_2(0)$ , this case cannot appear for this example.

**6.2. Boeing 747.** As the second example, we study the orbit closure hierarchy of a linearized nominal longitudinal model of a Boeing 747 considered in [51]. In our model we have joined nine inputs into five, which results in a model with 5 states, 6 outputs, and 5 inputs:

$$x = \begin{bmatrix} \delta q \\ \delta V_{TAS} \\ \delta \alpha \\ \delta \theta \\ \delta h_e \end{bmatrix} \begin{pmatrix} \text{pitch rate (rad/s)} \\ \text{true airspeed (m/s)} \\ \text{angle of attack (rad)} \\ \text{pitch angle (rad)} \\ \text{altitude (m)} \end{pmatrix}, \quad y = \begin{bmatrix} \delta \alpha \\ \delta \dot{V}_{TAS} \\ \delta \theta \\ \delta q \\ \delta V_z \\ \delta h_e \end{bmatrix} \begin{pmatrix} \text{angle of attack (rad)} \\ \text{acceleration (m/s}^2\text{)} \\ \text{pitch angle (rad)} \\ \text{pitch rate (rad/s)} \\ \text{vertical velocity (m/s)} \\ \text{altitude (m)} \end{pmatrix},$$

$$u = \begin{bmatrix} \delta_{ei} \\ \delta_{eo} \\ \delta_{ih} \\ \delta EPR_{1,4} \\ \delta EPR_{2,3} \end{bmatrix} \begin{pmatrix} \text{total inner elevator (rad)} \\ \text{total outer elevator (rad)} \\ \text{stabilizer trim angle (rad)} \\ \text{total thrust engine \#1 and \#4 (rad)} \\ \text{total thrust engine \#2 and \#3 (rad)} \end{pmatrix},$$

and the state-space matrices:

$$A = \begin{bmatrix} -0.4861 & 0.000317 & -0.5588 & 0 & -2.04 \cdot 10^{-6} \\ 0 & -0.0199 & 3.0796 & -9.8048 & 8.98 \cdot 10^{-5} \\ 1.0053 & -0.0021 & -0.5211 & 0 & 9.30 \cdot 10^{-6} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -92.6 & 92.6 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} -0.291 & -0.2988 & -1.286 & 0.0026 & 0.007 \\ 0 & 0 & -0.3122 & 0.3998 & 0.3998 \\ -0.0142 & -0.0148 & -0.0676 & -0.0008 & -0.0008 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & -0.0199 & 3.0796 & -9.8048 & 8.98 \cdot 10^{-5} \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -92.6 & 92.6 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.3122 & 0.3988 & 0.3988 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

These state-space matrices correspond to a Boeing 747 under straight-and-level flight at altitude 600  $m$  with speed 92.6  $m/s$ , flap setting at  $20^\circ$ , and landing gears up. The aircraft has mass = 317,000  $kg$  and the center of gravity coordinates are  $X_{cg} = 25\%$ ,  $Y_{cg} = 0$ , and  $Z_{cg} = 0$ .

The corresponding controllability pencil of the state-space system is of size  $5 \times 10$  and the observability pencil of size  $11 \times 5$ . First, let us consider the controllability pencil. Using StratiGraph the complete stratification of the orbit to a  $5 \times 10$  controllability pencil can be computed, which has 62 nodes and 108 edges. In our case, we are only interested to know the closest uncontrollable systems which can be reached by a perturbation of the system matrices. Instead of generating the complete stratification

we derive only the controllable and the nearest uncontrollable systems, starting with the controllability pencil given by the state-space matrices  $A$  and  $B$  above.

As in the previous example, we begin by determining the KCF of the controllability pair  $(A, B)$  which is  $2L_2 \oplus L_1 \oplus 2L_0$  with codimension 4. From the KCF (and BCF) we can see that the system is controllable with only three of the five input signals.<sup>2</sup>

Using the set of rules **A** and **C** in Table 5.1, the closure hierarchy around  $(A, B)$  can be determined. The resulting stratification graph is shown in Figure 6.3, where node 4:1 corresponds to the orbit which  $(A, B)$  belongs to. We now take the structural restrictions of  $A$  and  $B$  into consideration. By keeping all zeros and ones constant and choosing all free elements in  $A$  and  $B$  nonzero, it follows that the most generic orbit must have at least  $2L_0$  blocks; The number of  $L_0$  blocks is  $m - \text{rank}(A) = 5 - 3 = 2$ . This exclude  $\mathcal{O}(5L_1)$  and  $\mathcal{O}(L_2 \oplus 3L_1 \oplus L_0)$  from possible orbits and the most generic orbit is indeed the one  $(A, B)$  belongs to. The most degenerate orbit has KCF  $5L_1 \oplus J_2(\mu_1) \oplus 3J_1(\mu_2)$ , which is obtained by considering the system with all parameters set to zero. This orbit is however more degenerate than those of interest.

Using the stratification graph together with bounds on the distance to uncontrollability we can validate the robustness of the system. For a controllable pair  $(A, B)$ , the distance to uncontrollability [48] is defined as

$$\tau(A, B) = \min \left\{ \|\Delta A \quad \Delta B\| : (A + \Delta A, B + \Delta B) \text{ is uncontrollable} \right\},$$

where  $\|\cdot\|$  denotes the 2-norm or Frobenius norm. Equivalently,

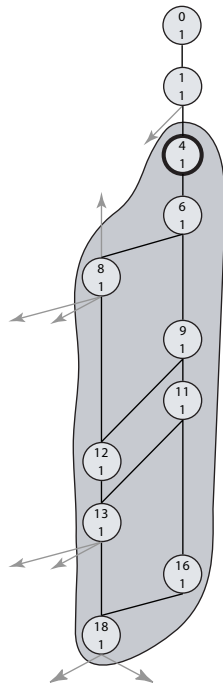
$$\tau(A, B) = \inf_{\lambda \in \mathbb{C}} \sigma_{\min} \left( \begin{bmatrix} A - \lambda I & B \end{bmatrix} \right),$$

where  $\sigma_{\min}(X)$  denotes the smallest singular value of  $X \in \mathbb{C}^{n \times (n+m)}$  [19]. Using the MATLAB implementation [47] of the methods presented in [34, 46], the distance to uncontrollability can be computed where  $\tau(A, B)$  is bounded within an interval  $(l, u]$  with any desired accuracy  $tol \geq u - l$ . For the above system the computed distance to uncontrollability is within  $(3.0323e-2, 3.0332e-2]$ , where  $tol = 10^{-5}$ .

Furthermore, using the technique presented in [22], the upper and lower bounds to all less generic controllability pairs shown in Figure 6.3 can be computed, see Table 6.1. The upper bounds are based on staircase regularizing perturbations, and the lower bounds are of Eckart-Young type and are derived from the matrix representations  $T_{(A,B)}$  (4.1) and  $T_{(A,C)}$  (4.2) of  $\tan(A, B)$  and  $\tan(A, C)$ , respectively. For the upper bounds, the implemented algorithm uses a naive approach to find a nearby matrix pair and the computed upper bounds are sometimes too conservative. However, we can observe that the above computed distance to uncontrollability is within the bounds of the uncontrollable systems with codimensions 8, 12, and 13.

Briefly, we also consider the  $11 \times 5$  observability pencil  $\mathbf{S}_O(\lambda)$  given by the above state-space matrices. This matrix pair has the KCF  $5L_1^T \oplus L_0^T$  with codimension 0, i.e., it is completely observable. Considering the structural restrictions of  $(A, C)$ , the most degenerate orbit possible has the KCF  $4L_1^T \oplus 2L_0^T \oplus J_1(\mu)$  with codimension 7. This can be seen by studying the matrix  $C$  with all parameters set to zero; At most two  $L_0^T$  blocks can exist,  $p - \text{rank}(C) = 5 - 3 = 2$ . Using the set of rules **E** (and **G**) in Table 5.1, the closure hierarchy shown in Figure 6.4 is derived.

<sup>2</sup>The other two inputs (corresponding to the  $L_0$  blocks) can be removed without loss of controllability. However, for safety reasons it is customary to have redundancy in the actuation system and the corresponding control surface in critical systems.



<b>Codimension 0</b>	1	$5L_1$
<b>Codimension 1</b>	1	$L_2 \oplus 3L_1 \oplus L_0$
<b>Codimension 4</b>	1	$2L_2 \oplus L_1 \oplus 2L_0$
<b>Codimension 6</b>	1	$L_3 \oplus 2L_1 \oplus 2L_0$
<b>Codimension 8</b>	1	$L_2 \oplus 2L_1 \oplus 2L_0 \oplus J_1(\mu)$
<b>Codimension 9</b>	1	$L_3 \oplus L_2 \oplus 3L_0$
<b>Codimension 11</b>	1	$L_4 \oplus L_1 \oplus 3L_0$
<b>Codimension 12</b>	1	$2L_2 \oplus 3L_0 \oplus J_1(\mu)$
<b>Codimension 13</b>	1	$L_3 \oplus L_1 \oplus 3L_0 \oplus J_1(\mu)$
<b>Codimension 16</b>	1	$L_5 \oplus 4L_0$
<b>Codimension 18</b>	1	$L_4 \oplus 4L_0 \oplus J_1(\mu)$

FIG. 6.3. Subgraph of the complete orbit stratification of a controllability pencil of size  $5 \times 10$ , where the grey area marks the possible structures for the Boeing 747 model. The node with codimension 4 represents the orbit to a system corresponding to a Boeing 747 under flight. The four nodes in the left-most branch of the graph represent the orbits of uncontrollable systems with one uncontrollable mode.

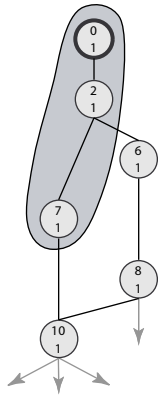
TABLE 6.1

Lower and upper bounds from the controllability pair  $(A, B)$  of a Boeing 747 under flight with KCF  $2L_2 \oplus L_1 \oplus 2L_0$  to the less generic orbits shown in Figure 6.3.

Imposed structure from $2L_2 \oplus L_1 \oplus 2L_0$	cod	Lower bound	Upper bound
$L_3 \oplus 2L_1 \oplus 2L_0$	6	$1.29e-4$	$4.02e-2$
$L_2 \oplus 2L_1 \oplus 2L_0 \oplus J_1(\mu)$	8	$4.33e-4$	1.0
$L_3 \oplus L_2 \oplus 3L_0$	9	$5.97e-4$	$1.59e-3$
$L_4 \oplus L_1 \oplus 3L_0$	11	$8.47e-4$	$1.59e-3$
$2L_2 \oplus 3L_0 \oplus J_1(\mu)$	12	$1.09e-3$	$2.48e-1$
$L_3 \oplus L_1 \oplus 3L_0 \oplus J_1(\mu)$	13	$1.33e-3$	$1.79e-1$
$L_5 \oplus 4L_0$	16	$1.78e-2$	$5.56e-1$
$L_4 \oplus 4L_0 \oplus J_1(\mu)$	18	$7.57e-2$	$5.56e-1$

**7. Conclusions.** We have derived the closure and cover conditions for orbits and bundles of matrix pairs, where the cover conditions are new results. In line with previous work on matrices and matrix pencils [17, 18], we have derived the stratification rules for matrix pairs, both for controllability pairs  $(A, B)$  and observability pairs  $(A, C)$ , in terms of coin moves.

The results are illustrated with two examples taken from real applications in systems and control. We show how the rules are used and how they provide qualitative information of a system, which together with distance information are useful for validating an LTI state-space system.



**Codimension 0**

1  $5L_1^T \oplus L_0^T$

**Codimension 2**

1  $L_2^T \oplus 3L_1^T \oplus 2L_0^T$

**Codimension 6**

1  $2L_2^T \oplus L_1^T \oplus 3L_0^T$

**Codimension 7**

1  $4L_1^T \oplus 2L_0^T \oplus J_1(\mu)$

**Codimension 8**

1  $L_3^T \oplus 2L_1^T \oplus 3L_0^T$

**Codimension 10**

1  $L_2^T \oplus 2L_1^T \oplus 3L_0^T \oplus J_1(\mu)$

FIG. 6.4. Subgraph of the complete orbit stratification of an observability pencil of size  $11 \times 5$ , where the grey area marks the possible structures for the Boeing 747 model. The node with codimension 0 represents the orbit to a system corresponding to a Boeing 747 under flight. The two nodes 7:1 and 10:1 represent the orbits of unobservable systems with one unobservable mode.

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