Term monad in monoidal biclosed categories

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Introduction.

A pair $(X, *)$ is a prequantale if X is a complete lattice and $*$ is binary operation on X satisfying the following distributive law:

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Does there exists a generalization of the term construction to **Sup** ?

A categorical formulation of the term construction in **Set**.

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Let Σ be a signature. A Σ-algebra is a pair (X, δ) where

- X is a set,
- $\delta = (\delta_n)_{n \in \mathbb{N}_0}$ is a sequence of maps $\Omega_n \times X^n \xrightarrow{\delta_n} X$ where X^n

denotes the n-th power of X w.r.t. the cartesian product and $X^0 = \{\cdot\}$.

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$$\bigsqcup_{n \in \mathbb{N}_0} \Omega_n \times X^n \xrightarrow{\delta} X.$$

A map $X \xrightarrow{h} Y$ is a Σ-homomorphism from a Σ-algebra (X, δ) to (Y, ε) if the following diagram is commutative:

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\begin{array}{ccc}
\coprod_{n \in \mathbb{N}_0} \Omega_n \times X^n & \xrightarrow{\bigoplus_{n \in \mathbb{N}_0} 1_{\Omega_n} \times h^n} & \coprod_{n \in \mathbb{N}_0} \Omega_n \times Y^n \\
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Free Σ-algebras exist!!

Usual term construction:

X = set of variables, Ω = set of operator symbol, $X \cap \Omega = \varnothing$.

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Aim: Formal term construction based on the data of **Set**.

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- $Z_0(X) = \bigcup_{k\in\mathbb{N}} Z_k(X)$ is the inductive limit of $(Z_k(X), e_{m\,k})_{k\in\mathbb{N}}$.

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&= \bigcup_{k\in\mathbb{N}} \Big(\coprod_{n\in\mathbb{N}_0} \Omega_n \times (Z_k(X) \sqcup X)^n\Big) \\
&= \bigcup_{k\in\mathbb{N}} Z_{k+1}(X) \\
&= Z_0(X)
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- There exists a bijective map $\displaystyle\bigsqcup_{n\in\mathbb{N}_0} \Omega_n \times T(X)^n \xrightarrow{\vartheta} Z_0(X)$

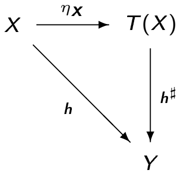
provided with the following properties

- $e_1 = \vartheta \circ \Big(\bigoplus_{n\in\mathbb{N}_0} 1_{\Omega_n} \times (\eta_X)^n\Big),$
- $e_{k+1} = \vartheta \circ \Big(\bigoplus_{n\in\mathbb{N}_0} 1_{\Omega_n} \times (e_k \oplus 1_X)^n\Big), \qquad k \in \mathbb{N}.$

$(T(X), j_0 \circ \vartheta)$ is the term Σ-algebra.

Theorem

Let Σ be a signature, X be a set and $(T(X), j_0 \circ \vartheta)$ be the term algebra. For every Σ-algebra (Y, δ) and for every map $X \xrightarrow{h} Y$ there exists a unique homomorphism $(T(X), j_0 \circ \vartheta) \xrightarrow{h^\sharp} (Y, \delta)$ making the following diagram commutative:



(E)

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- $$\begin{array}{ccccc}
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 & & \Big\downarrow {\scriptstyle \bigoplus\limits_{n \in \mathbb{N}_0} 1_{\Omega_n} \times (h^\sharp)^n} & & \Big\downarrow {\scriptstyle h^\sharp} \\
 & & \coprod\limits_{n \in \mathbb{N}_0} \Omega_n \times Y^n & \xrightarrow[\delta]{} & Y
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- The restriction of $h^\sharp$ to $Z_0(X)$ — i.e. $h^\sharp \circ j_0$ — is uniquely determined by h.

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- $h^\sharp$ is a homomorphism.

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- In this context:
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- Theorem. The forgetful functor from the category of Σ-algebras in the monoidal biclosed category $\mathcal{C}$ has a left adjoint functor.

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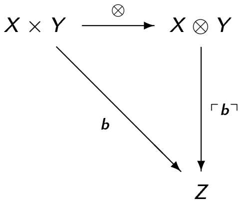
- 

where $\otimes$ denotes the universal bimorphism.

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- Result: Every complete lattice X generates a free prequantale.

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$$\begin{array}{ccc} (X, f_X) \otimes (Y, f_Y) & \xrightarrow{1_X \times \ulcorner b \urcorner_r} & (X, f_X) \otimes (Z^X, g_r) \\ & {\scriptstyle b} \searrow & \downarrow {\scriptstyle ev} \\ & & (Z, f_Z) \end{array} \qquad [\ulcorner b \urcorner_r(y)](x) \;=\; b(x, y).$$

The internal hom-objects are given as follows:

- $[(X, f_X), (Z, f_Z)]_r \;=\; (Z^X, g_r)$, where
- $g_r(\alpha) \;=\; \bigwedge_{x \in X} f_X(x) \searrow f_Z(\alpha(x)), \quad \alpha \in Z^X.$

- $[(X, f_X), (Z, f_Z)]_\ell \;=\; (Z^X, g_\ell)$, where
- $g_\ell(\alpha) \;=\; \bigwedge_{x \in X} f_Z(\alpha(x)) \swarrow f_X(x), \quad \alpha \in Z^X.$

The following diagrams are commutative:

- $$\begin{array}{ccc} (X, f_X) \otimes (Y, f_Y) & \xrightarrow{1_X \times \ulcorner b \urcorner_r} & (X, f_X) \otimes (Z^X, g_r) \\ & \searrow^{b} & \downarrow{\scriptstyle ev} \\ & & (Z, f_Z) \end{array}$$ $$[\ulcorner b \urcorner_r(y)](x) \;=\; b(x, y).$$

- $$\begin{array}{ccc} (X, f_X) \otimes (Y, f_Y) & \xrightarrow{\ulcorner b \urcorner_\ell \times 1_Y} & (Z^Y, g_\ell) \otimes (Y, f_Y) \\ & \searrow^{b} & \downarrow{\scriptstyle ev} \\ & & (Z, f_Z) \end{array}$$ $$[\ulcorner b \urcorner_\ell(x)](y) \;=\; b(x, y).$$

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- $(f_X)^\sharp$ is the unique extension of $X \xrightarrow{f_X} (Q, *)$ to a homomorphism.
- This construction turns the binary operation $\bullet$ of $X^\sharp$ into the binary operation in the sense of Goguen's category:

$$(X^\sharp, (f_X)^\sharp) \otimes (X^\sharp, (f_X)^\sharp) \xrightarrow{\ \bullet\ } (X^\sharp, (f_X)^\sharp).$$